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Bakerian Lecture: The Structure and Opacity of a Stellar Atmosphere

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X. BAKERIAN LECTURE.—*The Structure and Opacity of a Stellar Atmosphere.*

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CONTENTS.

1. *The general problem.*—"Stellar atmosphere" is the name given loosely to the outer portions of a star. The stellar atmosphere is divided observationally into three superincumbent layers, named the photospheric layers, the reversing layer and the chromosphere, in order of increasing level. The boundaries between these are only roughly defined, but broadly speaking the photospheric layers give rise to the continuous spectrum of the star, the reversing layer to the absorption-line spectrum and the chromosphere (when seen edgeways) to the flash spectrum.* Mathematical analysis of the way in which gaseous material comprising the outer portions of a star may be expected to thin out into space confirms this threefold division. It also brings to light

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* Actually only so studied for the Sun, but the term is useful generally.

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certain dynamical and thermal characteristics of the three layers. For example a definite temperature gradient in the photospheric layers shades off into an approximately isothermal state in the chromosphere; "local thermodynamic equilibrium" in the photospheric layers shades off into "monochromatic radiative equilibrium" in the upper chromosphere; and a somewhat unimportant general radiation pressure in the photospheric layers augments to a strong selective radiation-pressure in the reversing layer and chromosphere. The reversing layer is in most cases the transition layer. Assumptions valid for either photospheric layers or chromosphere separately cease to be so near their upper and lower boundaries respectively and so far it has not been possible to give a treatment which accurately deals with the regions of transition. In the present lecture it is proposed to consider chiefly the photospheric layers and the reversing layer. For these regions the dominant need is the determination of the general opacity—the fogginess—for this determines the depth we see into the star and so the pressures, densities, etc., at which the observed spectral phenomena originate.

The abstract problem of the stellar atmosphere may be stated as follows. For many purposes the curvature of the outer regions of a star may be neglected and we consider only material stratified in parallel planes. The material is subject to (*a*) a gravitational field of acceleration g, (*b*) a net flux of energy of amount πF per unit area, incident on it from below and emergent into space above. This is determined by the evolution of energy in the interior of the star. The amount of energy actually incident on the atmospheric layers from below exceeds πF, but a portion is re-radiated downwards by the atmospheric layers, πF being the net amount passing through. If the atmospheric layers are in a steady state* there is no accumulation of energy, and the net amount of energy crossing any surface of stratification is equal to that crossing any parallel surface, namely πF. The quantity F itself is the mean value of the emergent *intensity* of radiation at any point, or, what is the same thing, the mean intensity of radiation over the stellar disc. The abstract problem is:—Given the two parameters g and F, and given also the ultimate chemical composition of the material, to determine the distribution of temperature, pressure, density, ionization and chemical composition in the layers, and to determine also the complete intensity-distribution both in angle and in frequency, of the emergent radiation. The practical problem is to some extent the converse one of inferring the temperature and other physical quantities from the observed emergent radiation, *i.e.*, from the observed spectra, measured if possible spectro-photometrically. In many cases we do not know either g or F, and these also may have to be determined from the observed spectra.

The dependence on two parameters g and F (assuming similar chemical composition) corresponds to the observed two-fold classification of stellar spectra.

On the one hand we have the grand sequence of types of spectra, principally comprised in the Draper classes M, K, G, F, A, B, O, and correlated with systematic changes of colour. This is taken to correspond to changes in the variable F, or, as is more

* In this lecture only steady states are considered.

usually stated, to change in "effective temperature." A full radiator emitting energy πF per cm.^{2} would have a temperature T_1 given by $\pi F = \sigma T_1^4$, where σ is STEFAN's constant; T_1, defined in this way, is called the "effective temperature" of the star. Strictly speaking, only for a very few stars are the effective temperatures known. If L is the absolute bolometric brightness of a star, l its apparent bolometric brightness, r_1 its radius, R its distance, α its angular diameter, then $L \propto r_1^2 T_1^4$, $l \propto L/R^2$, $\alpha \propto r_1/R$, whence $T_1 \propto l^{\frac{1}{4}} \alpha^{-\frac{1}{2}}$. Thus a determination of effective temperature requires a brightness measure combined with a measure of angular diameter. For the few stars for which α has been determined experimentally the effective temperature T_1 agrees with the "colour temperature" or temperature determined by the *shape* of the energy-curve for the continuous spectrum. The exact relation between colour-temperature and effective temperature remains for future investigation,* but provisionally it is assumed that the observed correlation of type of spectrum with colour-temperature indicates a correlation of type with effective temperature.

On the other hand for stars of given type there are smaller but well-marked variations of spectrum—variations in relative intensity of pairs of lines—which have been found to be correlated with absoute luminosity.† It is generally agreed‡ that this correlation must in fact be a correlation of spectrum with the second parameter, surface gravity g.

The explanation of the observed correlations in its broad outline is due to SAHA,§ who proposed the theory of the high-temperature ionization of the elements and traced its consequences. Variations of spectral type are due to variations of ionization and excitation consequent on variations of temperature—actual temperature, not necessarily effective temperature. Variations of spectrum inside a given spectral type are due to variations of ionization consequent on variations of pressure, which in turn are consequences of variations of surface gravity. The researches summarized in this lecture are attempts to obtain definite quantitative laws for the calculation of intensities of spectral lines as functions of the two fundamental parameters F (or T_1) and g.

The quantitative applications have resulted first in a knowledge of the pressures in stellar atmospheres, secondly, in a knowledge of the coefficient of general opacity of the material forming the photospheric layers, and they bid fair to give ultimately a

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* A determination of effective temperature might be possible for an eclipsing binary of ascertained surface brightness and parallax. The problem of the low colour-temperatures of many early-type stars might be investigated in this way, by comparing the observed colour-temperature (deduced from *shape* of energy-curve) with the observed effective temperature (deduced from the absolute values of the ordinate of the energy-curve).

† ADAMS and KOHLSCHÜTTER, 'Astrophys. J.,' vol. 40, p. 385 (1914); ADAMS and JOY, *ibid.*, vol. 46, p. 313 (1917); etc.

‡ *e.g.*, PANNEKOEK, 'Bull. Ast. Neth.,' vol. 1 (No. 19), p. 107 (1922); 'Observatory,' vol. 46, p. 304 (1923).

§ 'Roy. Soc. Proc.,' A, vol. 99, p. 135 (1921). 'Phil. Mag., vol. 40, pp. 472, 809 (1920). A particular case of high-temperature ionization had been considered earlier by LINDEMANN, 'Phil. Mag.,' vol. 38, p. 676 (1919).

complete scheme of the chemical composition (in proportion by mass) of stellar atmospheres. Recent progress depends on a combination of a fundamental idea due to STEWART and UNSÖLD* with the ideas of SAHA, along lines of investigation originated by Sir ARTHUR SCHUSTER and SCHWARZSCHILD† on the thermal structure of stellar atmospheres. STEWART and UNSÖLD have developed the idea of the breadth and intensity of a spectral line as arising from the " number of atoms above the photosphere, per cm.^{2}." Development of the ideas of SCHUSTER and SCHWARZSCHILD permits us to give a precise meaning to the phrase " above the photosphere. Refinements of the methods of calculation originated by SAHA permit the calculation of the " number of atoms in the column " as a function of temperature and pressure at the base of the column. It appears that the fundamental desideratum is the intrinsic opacity of the photospheric material, as this determines the pressure at the base of the column. Observations of maxima of lines in the stellar sequence of spectra permit the determination of the opacity without introducing UNSÖLD's formula for the atomic (selective) scattering coefficient. Determination of the absolute value of the coefficient of opacity of photospheric material sets the way open for calculations of " numbers of atoms above the photosphere " in terms of the absolute proportions by mass of the constituents. Comparison of the calculated values with those derived by observation of line-contours and reduction by UNSÖLD's method gives either a check on UNSÖLD's formula for the atomic scattering coefficient or a determination of an absolute abundance (proportion by mass). The observations of Miss PAYNE‡ and her co-workers at the Harvard College Observatory give precisely the information required to test the whole theory, which is in general confirmed. They indicate the order of magnitude of the proportion of the element observed (calcium) in stellar atmospheres, and will probably eventually determine " effective gravity " as compared with true gravity, *i.e.*, will determine the importance of selective radiation pressure.

I.—*Analysis of thermal structure of stellar atmospheres.*

2. SCHWARZSCHILD's *problem.*—Associated with the names of SCHWARZSCHILD and SCHUSTER are two idealised problems which have had great influence on the development of the theory. Mathematically they are completely equivalent. It is necessary to state the solutions of these two problems.

SCHWARZSCHILD's problem is that of finding the temperature distribution for material in radiative equilibrium possessing an absorption coefficient independent of wave-length, it being given that the material is traversed by a net-flux πF. The material is supposed stratified in parallel planes, and extends to infinity in the direction

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* STEWART, 'Astrophys. J.,' vol. 59, p. 30 (1924); UNSÖLD, 'Z. Physik, vol. 44. p. 793 (1927), and vol. 46, p. 765 (1928).

† SCHUSTER, 'Astrophys. J.,' vol. 21, p. 1 (1905); SCHWARZSCHILD, 'Gött. Nach.," p. 41 (1906).

‡ PAYNE and HOGG, 'Harvard Circ.,' p. 334 (1928); PAYNE and WILLIAMS, 'M.N.R.A.S.,' vol. 89, p. 526 (1929).

from which the net flux is coming. If $B = (\sigma/\pi) T^4$, the problem is embodied by the equations

$$\cos\theta \frac{dI(\tau, \theta)}{d\tau} = I(\tau, \theta) - B, \quad \quad (1)$$

$$\int I(\tau, \theta)\, d\omega = 4\pi B, \quad \quad (2)$$

$$d\tau = \kappa\rho\, dh, \quad \quad (3)$$

where h is the depth measured inwards, ρ is the density, κ the absorption coefficient, τ is the optical depth and $I(\tau, \theta)$ is the intensity of radiation in a direction making θ with the outward normal to the planes of stratification. Equation (1) is the "equation of transfer,"* equation (2) the equation of radiative equilibrium. The problem is to determine B as a function of τ. The function B can be shown† to be included amongst the solutions of the integral equation

$$B(\tau) = \tfrac{1}{2}\int_0^\tau B(t)\, Ei(\tau - t)\, dt + \tfrac{1}{2}\int_\tau^\infty B(t)\, Ei(t - \tau)\, dt \quad . \quad . \quad (4)$$

where

$$Ei(x) = \int_1^\infty \frac{e^{-\mu x}}{\mu}\, d\mu.$$

This, however, does not put in evidence the net flux πF. To exhibit the dependence of B on F, and to pick out the solution of (4) which we require, we replace (2) by the equivalent equation‡ expressing the constancy of net flux,

$$\int I(\tau, \theta)\cos\theta\, d\omega = \pi F. \quad \quad (5)$$

It is then found that B satisfies§

$$F = 2\int_\tau^\infty B(t)\, Ei_2(t - \tau)\, dt - 2\int_0^\tau B(t)\, Ei_2(\tau - t)\, dt, \quad \quad (6)$$

where

$$Ei_2(x) = \int_1^\infty \frac{e^{-\mu x}}{\mu^2}\, d\mu.$$

An approximate solution of (6), satisfying it exactly for $\tau = 0$ and for τ large, is||

$$B = \tfrac{1}{2} F\left(1 + \tfrac{3}{2}\tau\right). \quad \quad (7)$$

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* The term is due to Sir James Jeans, 'M.N.R.A.S.,' vol. 78, p. 28 (1917).

† 'M.N.R.A.S.,' vol. 81, p. 361 (1921).

‡ Equation (5) follows from (1) by multiplying by $d\omega$, integrating over the solid angle 4π, using (2), and integrating with respect to τ. It can also be written down from first principle, by the conditions of the problem.

§ Equation (6) does not appear to have been given before. It is readily derived by solving (1) for $I(\tau, \theta)$ and inserting in (5), and then inverting the order of integration.

|| 'M.N.R.A.S.,' vol. 81, p. 361 (1921); 'Phil. Trans.,' A, vol. 223, p. 202 (1922).

An approximate solution of SCHWARZSCHILD'S problem is accordingly

$$T^4 = \tfrac{1}{2} T_1^4 (1 + \tfrac{3}{2} \tau). \qquad (8)$$

Associated with this temperature distribution is the law of distribution of the emergent radiation

$$I(0, \theta) = \tfrac{1}{2} F (1 + \tfrac{3}{2} \cos \theta). \qquad (9)$$

Relations (7), (8) and (9) are solutions of SCHWARZSCHILD'S problem sufficiently accurate for ordinary purposes. (8) implies a boundary temperature $T_0 = 2^{-\frac{1}{4}} T_1 = 0 \cdot 84\, T_1$. More accurate investigations show that there is a sudden drop near the boundary, $dB(\tau)/d\tau$ becoming infinite like $\log(1/\tau)$, and the boundary temperature is very close to $0 \cdot 813\, \tau$. The law of darkening (9), corresponding to a coefficient of darkening $\frac{3}{5}$, is fairly well confirmed by observations of the integrated radiation at various points of the sun's disc; observed departures from (9) can be shown to be accounted for by departures from the ideal conditions postulated in SCHWARZSCHILD'S problem, namely the absence of line absorption.* The verification of (9) is observational proof of the existence of radiative equilibrium in the sun's photospheric layers.

The utility of the solution to SCHWARZSCHILD'S problem is that it relates actual temperatures to the effective temperature.

3. SCHUSTER'S *Problem.*—SCHUSTER'S problem can be stated as follows. A layer of gas, capable of scattering monochromatic radiation of frequency ν with scattering coefficient s_ν, is placed in front of a bright background radiating with a given flux πG_ν; given the optical thickness σ_ν of the scattering material, to determine the emergent flux.

The solution is readily derived from the solution to SCHWARZSCHILD'S problem. The equation of transfer of ν-radiation is

$$\cos \theta \frac{dI_\nu(\sigma_\nu, \theta)}{d\sigma_\nu} = I_\nu - \int I_\nu \frac{d\omega}{4\pi}, \qquad (11)$$

where

$$d\sigma_\nu = s_\nu \rho\, dh.$$

Thus the law (11) governing the monochromatic radiation in SCHUSTER'S problem is the same as the combined laws (1) and (2) of SCHWARZSCHILD'S problem. The mean intensity $\int I_\nu(\sigma_\nu, \theta)\, d\omega/4\pi$ therefore follows the law

$$\int I_\nu(\sigma_\nu, \theta) \frac{d\omega}{4\pi} = \tfrac{1}{2} F_\nu (1 + \tfrac{3}{2} \sigma_\nu), \qquad (9)$$

but in SCHUSTER'S problem F_ν is so far unknown. Now in SCHWARZSCHILD'S problem the outward and inward fluxes at τ are readily found to be approximately

$$\pi F_+ = \pi F (1 + \tfrac{3}{4} \tau)$$
$$\pi F_- = \pi F (\tfrac{3}{4} \tau),$$

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* 'Observatory,' vol. 51, p. 88 (1928).

the difference being πF. Replacing πF_+ by πG_ν, the given incident flux in SCHUSTER's problem, and replacing τ by σ_ν we have the answer to SCHUSTER's problem in the form

$$\frac{F_\nu}{G_\nu} = \frac{1}{1+\frac{3}{4}\sigma_\nu}, \quad \ldots \ldots \ldots \ldots \quad (10)$$

where here σ_ν denotes the optical thickness of the complete layer. Formula (10) is valid provided σ_ν is not too small. It is independent of the distribution in direction of the incident radiation provided its total flux is πG_ν. The emergent radiation follows law (9) as in SCHWARZSCHILD's problem. Formula (10) gives the intensity-ratio in an "absorption line" produced by monochromatically scattering atoms placed in front of a radiating surface.

4. *Relation between* SCHWARZSCHILD's *Problem and* SCHUSTER's *Problem.*—We have here two idealisations neither of which corresponds completely to a stellar atmosphere. SCHWARZSCHILD's problem ignores line absorption but probably gives a correct indication of the temperature distribution. SCHUSTER's problem is appropriate to the investigation of line absorption, but introduces the concept of a definite photospheric surface. Quantum absorption switches followed by emissions of the same frequency are formally equivalent to scattering without change of frequency, so that SCHUSTER's scattering coefficient s_ν is easily expressed in terms of an Einstein B coefficient. But if we idealise the stellar atmosphere to the conditions of SCHUSTER's problem we must calculate where to put the photospheric radiating surface. In a real stellar atmosphere the upper layers contributing the line absorption must shade continuously into the deeper layers contributing the general opacity.

If in SCHUSTER's formula (10) we consider s_ν as a sensitive function of ν in the neighbourhood of a characteristic frequency ν_0, zero save in this immediate neighbourhood, then F_ν/G_ν is unity save near $\nu = \nu_0$, and as ν approaches ν_0 and σ_ν increases, F_ν/G_ν decreases. We have thus a fair representation of a "line-contour" if we regard G_ν as the intensity of the continuous spectrum between the lines. Observation of the ratio $r_\nu = F_\nu/G_\nu$ then determines σ_ν. If α_ν is the *atomic* scattering coefficient, N the number of atoms per cm.^{2} down to the photospheric radiating surface, then

$$\sigma_\nu = N\alpha_\nu. \quad \ldots \ldots \ldots \ldots \quad (11)$$

Hence if α_ν is known from physical theory, N can be determined from the observed line-contour. This is the method of UNSÖLD. But it does not follow from this that N will be independent of ν, for the observed shape may not agree exactly with the theoretical α_ν. In fact until we know how deep to place the photospheric surface there is considerable uncertainty as to what we mean by N. It may be different for different parts of the line-contour. We require a fusion of the SCHUSTER and SCHWARZSCHILD models.

5. *Solution for Simultaneous Scattering and Absorption.*—Let us follow the SCHWARZSCHILD model in having no definite background surface; the material is to extend inwards

indefinitely. Let us follow the SCHUSTER model in retaining monochromatic scattering. The equation of transfer, for a scattering coefficient s_ν and an absorption coefficient κ_ν now takes the form

$$\cos\theta \frac{dI_\nu}{\rho dh} = (\kappa_\nu + s_\nu) I_\nu - \kappa_\nu B_\nu - s_\nu \int I_\nu \frac{d\omega}{4\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

$$B_\nu = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equation (12) follows from purely macroscopic considerations, assuming the material is in local thermodynamic equilibrium. It can however be derived—or rather a similar equation, from entirely different considerations—from the quantum theory of emission and absorption as modified by collisions.* It is not necessary to assume local thermodynamic equilibrium, and the function B_ν makes its appearance not from an appeal to KIRCHHOFF'S or PLANCK'S law but from an application of the principle of detailed balancing to enclosure conditions. All that is necessary to assume is a Maxwellian velocity distribution; the effect of collisions in exciting and de-exciting atoms is then the same as in an enclosure, though enclosure conditions are not assumed in calculating the relative numbers of normal and excited atoms. The ratio κ_ν/s_ν is replaced by a ratio in which the denominator is the EINSTEIN B coefficient and the numerator measures the relative importance of collisions. It follows that near the surface, where the density is low and collisions are negligible, the terms in κ_ν in (12) may be neglected in comparison with those in s_ν; in the deep interior, where collisions are all important, the terms in s_ν in (12) may be neglected in comparison with those in κ_ν. Near the surface the conditions are those of pure monochromatic scattering—SCHUSTER'S problem; hence the solution of SCHUSTER'S problem is strictly apposite for the chromosphere. In the deep interior (photospheric layers) the conditions are those of local thermodynamic equilibrium—SCHWARZSCHILD'S problem. It is evident that in writing down (12) we have successfully fused the two problems.

It would be desirable to solve (12) with κ_ν/s_ν an increasing function of density (approximately, $\kappa_\nu/s_\nu \propto \rho$). This has not yet been accomplished. Failing this we take a mean value for κ_ν/s_ν, and we expect the results to be satisfactory for the transition region—the reversing layer. Actually the significant ratio is $\kappa_\nu/(\kappa_\nu + s_\nu)$ which varies from 0 to 1 between boundary and interior, so the assumption of a mean value for $\kappa_\nu/(\kappa_\nu + s_\nu)$ is not too violent an approximation. Formally the result is equivalent to taking κ_ν and s_ν independent of h in (12).

Following a procedure due to EDDINGTON,† we now write

$$J_\nu = \int I_\nu \frac{d\omega}{4\pi}, \quad \pi F_\nu = \int I_\nu \cos\theta \, d\omega, \quad L_\nu = \int I_\nu \cos^2\theta \frac{d\omega}{4\pi}, \quad . \quad . \quad . \quad (14)$$

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* 'M.N.R.A.S.,' vol. 88, p. 493 (1928).

† 'Internal Constitution of the Stars,' p. 322 (1926).

and put

$$\rho\, dh = dm. \qquad (15)$$

J_ν, F_ν, L_ν are functions of the depth h, or of m. Then (12) on multiplication in turn by 1 and $\cos\theta$ and integration over all directions gives

$$\frac{d}{dm}\left(\tfrac{1}{4}F_\nu\right) = \kappa_\nu\left(J_\nu - B_\nu\right), \qquad (16)$$

$$\frac{d}{dm} L_\nu = \left(\kappa_\nu + s_\nu\right) \tfrac{1}{4}F_\nu. \qquad (17)$$

Introduce as an approximation

$$L_\nu = \tfrac{1}{3}J_\nu, \qquad (18)$$

which holds good exactly if $I_\nu(\theta)$ is expansible in odd powers of $\cos\theta$ in $0 \leq \theta \leq \pi$. This condition is not fulfilled at the boundary, but approximation (18), which avoids the occurrence of integral equations, is found to be of an accuracy comparable with the solutions of SCHWARZSCHILD's and SCHUSTER's problems already quoted. Eliminating F_ν we find

$$\tfrac{1}{3}\frac{d^2}{dm^2} J_\nu = \kappa_\nu\left(\kappa_\nu + s_\nu\right)\left(J_\nu - B_\nu\right). \qquad (19)$$

Let $\bar{\kappa}$ and $\bar{s}$ denote certain mean values of κ_ν and s_ν through the spectrum which will be defined later. Put

$$\frac{\kappa_\nu + s_\nu}{\bar{\kappa} + \bar{s}} = n_\nu, \qquad \frac{\kappa_\nu}{\kappa_\nu + s_\nu} = \xi_\nu^{\,2}, \qquad (20)$$

$$\left(\bar{\kappa} + \bar{s}\right) dm = d\tau. \qquad (21)$$

Then (19) becomes

$$\tfrac{1}{3}\frac{d^2 J_\nu}{d\tau^2} = n_\nu^{\,2}\xi_\nu^{\,2}\left(J_\nu - B_\nu\right). \qquad (22)$$

Equation (22) has to be solved with the two boundary conditions (1) for τ large, J_ν is at most of the order of magnitude of B_ν, (2) at the boundary $\tau = 0$ the inward radiation is zero; this latter condition is equivalent, to a degree of approximation consistent with (18), to the condition

$$\tfrac{1}{4}F_\nu(0) = \tfrac{1}{2}J_\nu(0). \qquad (23)$$

After some algebra we arrive at the results

$$J_\nu(\tau) = \frac{\dfrac{\xi_\nu}{\sqrt{3}} - \dfrac{1}{2}}{\dfrac{\xi_\nu}{\sqrt{3}} + \dfrac{1}{2}}\,\tfrac{1}{2}e^{-n_\nu\xi_\nu\tau\sqrt{3}} \int_0^\infty B_\nu(t)\, e^{-n_\nu\xi_\nu t\sqrt{3}}\, n_\nu\xi_\nu\sqrt{3}\, dt$$

$$+ \tfrac{1}{2}e^{+n_\nu\xi_\nu\tau\sqrt{3}} \int_\tau^\infty B_\nu(t)\, e^{-n_\nu\xi_\nu t\sqrt{3}}\, n_\nu\xi_\nu\sqrt{3}\, dt + \tfrac{1}{2}e^{-a_\nu\xi_\nu t\sqrt{3}} \int_0^\tau B_\nu(t)\, e^{n_\nu\xi_\nu t\sqrt{3}}\, n_\nu\xi_\nu\sqrt{3}\, dt, \qquad (23)$$

$$\mathrm{F}_\nu(\tau) = \frac{2\xi_\nu}{\sqrt{3}}\left[\frac{\frac{1}{2}-\dfrac{\xi_\nu}{\sqrt{3}}}{\frac{1}{2}+\dfrac{\xi_\nu}{\sqrt{3}}}\, e^{-n_\nu\xi_\nu\tau\sqrt{3}}\int_0^\infty \mathrm{B}_\nu(t)\, e^{-n_\nu\xi_\nu t\sqrt{3}}\, n_\nu\xi_\nu\sqrt{3}\, dt \right.$$
$$\left. + e^{n_\nu\xi_\nu\tau\sqrt{3}}\int_\tau^\infty \mathrm{B}_\nu(t)\, e^{-n_\nu\xi_\nu t\sqrt{3}}\, n_\nu\xi_\nu\sqrt{3}\, dt - e^{-n_\nu\xi_\nu\tau\sqrt{3}}\int_0^\tau \mathrm{B}_\nu(t)\, e^{n_\nu\xi_\nu t\sqrt{3}}\, n_\nu\xi_\nu\sqrt{3}\, dt\right]. \quad (24)$$

We have not yet determined the temperature distribution; we have not yet in fact imposed the condition of constant net flux for the integrated radiation of all frequencies. To make progress at this point we will assume that $\mathrm{B}_\nu(t)$ can be replaced by its two-term TAYLOR expansion

$$\mathrm{B}_\nu(t) = a_\nu + b_\nu t, \quad \ldots\ldots\ldots\ldots \quad (25)$$

where

$$a_\nu = \frac{2h\nu^3}{c^2}\cdot\frac{1}{e^{h\nu/k\mathrm{T}_0}-1}, \quad \ldots\ldots\ldots\ldots \quad (26)$$

$$b_\nu = \frac{2h\nu^3}{c^2}\cdot\frac{e^{h\nu/h\mathrm{T}_0}}{(e^{b\nu/k\mathrm{T}_0}-1)^2}\,\frac{h\nu}{k\mathrm{T}_0}\,\frac{1}{\mathrm{T}_0}\left(\frac{d\mathrm{T}}{dt}\right)_0. \quad \ldots\ldots\ldots \quad (27)$$

It will be found that this approximation preserves all the essential features of the investigation. Equation (23) and (24) now give

$$\mathrm{J}_\nu(\tau) = a_\nu + b_\nu\tau + \frac{\sqrt{3}}{2}\,\frac{\frac{2}{3}\dfrac{b_\nu}{n_\nu} - a_\nu}{\xi_\nu + \dfrac{\sqrt{3}}{2}}\, e^{-n_\nu\xi_\nu\tau\sqrt{3}}, \quad \ldots\ldots\ldots \quad (28)$$

$$\mathrm{F}_\nu(\tau) = \tfrac{4}{3}\frac{b_\nu}{n_\nu} + 2\xi_\nu\,\frac{a_\nu - \frac{2}{3}\dfrac{b_\nu}{n_\nu}}{\xi_\nu + \dfrac{\sqrt{3}}{2}}\, e^{-n_\nu\xi_\nu\tau\sqrt{3}}. \quad \ldots\ldots\ldots\ldots \quad (29)$$

We now impose the condition of constant net flux, in the form

$$\int_0^\infty \mathrm{F}_\nu(\tau)\, d\nu = \mathrm{F}. \quad \ldots\ldots\ldots\ldots\ldots \quad (30)$$

All we can hope for in accordance with the general tenour of our approximations is that (30) shall be satisfied exactly for $\tau = 0$ and for τ large. For τ large the exponential term in (29) vanishes and we require therefore

$$\tfrac{4}{3}\int_0^\infty \frac{b_\nu}{n_\nu}\, d\nu = \mathrm{F}. \quad \ldots\ldots\ldots\ldots\ldots \quad (31)$$

If we now put

$$\mathrm{B}(\tau) = \int_0^\infty \mathrm{B}_\nu\, d\nu = \int_0^\infty (a_\nu + b_\nu\tau)\, d\nu = a + b\tau,$$

and define $\bar{\kappa} + \bar{s}$ by the relation*

$$\frac{b}{\bar{\kappa} + \bar{s}} = \int_0^\infty \frac{1}{\kappa_\nu + s_\nu} b_\nu \, d\nu, \quad \ldots \ldots \quad (32)$$

relation (31) gives

$$b = \tfrac{3}{4}\mathrm{F}. \quad \ldots \ldots \quad (33)$$

This determines $(1/\mathrm{T}_0)\,(d\mathrm{T}/d\tau)_0$.

The satisfaction of (30) at $\tau = 0$ requires

$$\int_0^\infty \frac{\xi_\nu}{\xi_\nu + \frac{\sqrt{3}}{2}} \left(a_\nu - \tfrac{2}{3}\frac{b_\nu}{n_\nu}\right) d\nu = 0. \quad \ldots \ldots \quad (34)$$

This determines the boundary temperature T_0, which clearly depends on the behaviour of ξ_ν as a function of ν.† It is equally clear, however, that the order of magnitude of T_0 is independent of the absolute values of ξ_ν, and if this is constant or varies only slowly through the spectrum or has its rapid variation confined to small ranges of ν, (34) reduces approximately to

$$\int_0^\infty a_\nu \, d\nu = \tfrac{2}{3}\int_0^\infty \frac{b_\nu}{n_\nu} d\nu,$$

or by (31) to

$$a = \tfrac{1}{2}\mathrm{F}. \quad \ldots \ldots \quad (35)$$

Hence

$$\mathrm{B}(\tau) = \tfrac{1}{2}\mathrm{F}\,(1 + \tfrac{3}{2}\tau) \quad \ldots \ldots \quad (36)$$

and

$$\mathrm{T}^4 = \tfrac{1}{2}\mathrm{T}_1^{\,4}\,(1 + \tfrac{3}{2}\tau), \quad \ldots \ldots \quad (36\mathrm{A})$$

as in SCHWARZSCHILD's problem. This result is well known for the case when κ_ν and s_ν are independent of ν. We now see that it is true in general (apart from possible modifications in the boundary temperature) provided τ is calculated from the ROSSELAND mean sum of the scattering and absorption coefficients.

We can make two important applications of (29).

6. *Selective Radiation Pressure.*—The first is to selective radiation pressure, which inside a spectral interval $\Delta\nu$ on a layer $\rho\, dh$ gives rise to a force per unit area

$$\pi \mathrm{F}_\nu(\tau)\, \Delta\nu\, (k_\nu + s_\nu)\, \rho\, dh.$$

At the boundary $\tau = 0$ this will be found to reduce to

$$\pi\, \Delta\nu\, \rho\, dh \frac{2}{\xi_\nu + \frac{\sqrt{3}}{2}} \left[\mathrm{B}_\nu(0)\, k_\nu^{\frac{1}{2}} (k_\nu + s_\nu)^{\frac{1}{2}} + \frac{1}{\sqrt{3}} \left(\frac{d\mathrm{B}_\nu}{\rho dh}\right)_0\right]. \quad \ldots \quad (37)$$

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* This is an example of ROSSELAND's theorem, 'M.N.R.A.S.,' vol. 84, p. 525 (1924).

† It would be interesting to study T_0 as given by (34) for various forms of ξ_ν.

As τ increases it rapidly approaches a value which will be found to be

$$\pi\ \Delta\nu\ \rho\, dh\, \tfrac{4}{3}\left(\frac{dB_\nu}{\rho dh}\right). \qquad \qquad (38)$$

In expression (38) the selective effect of radiation pressure has entirely disappeared ; at great depths the effect of a large opacity $(\kappa_\nu + s_\nu)$ is neutralised by a small F_ν. But in (37) the term is $\kappa_\nu^{\frac{1}{2}}(\kappa_\nu + s_\nu)^{\frac{1}{2}}$ will become large if κ_ν and s_ν become large, whilst $B_\nu(0)$ is unaffected ; at the surface selective radiation pressure exists and may become very large. It is this phenomenon which gives rise to the existence of the chromosphere.*

7. *Determination of Equivalent Photospheric Surface.*—The second application is one of our main objects, namely to determine the optical depths at which absorption lines may be said to originate. We will first obtain the intensity-ratio (line/continuous spectrum). The absorption coefficient κ_ν represents the degree to which energy is truly absorbed, *i.e.*, available for conversion into other frequencies and not merely held pending re-radiation in the same frequency. It includes the part played by the photo-electric effect in giving rise to the continuous spectrum—it includes the effect of every kind of collision. We wish our formulæ to hold both for line absorption and for the continuous spectrum between the lines. We will therefore assume κ_ν to be constant throughout the spectrum, and denote it simply by κ. We will take s_ν to be zero save in the vicinity of a line, and hence we may take $\bar{s} = 0$. In the vicinity of a line we have then

$$n_\nu = \frac{\kappa + s_\nu}{\kappa}, \qquad \xi_\nu^2 = \frac{\kappa}{\kappa + s_\nu},$$

whence

$$n_\nu = 1/\xi_\nu^2.$$

Then from (29)

$$F_\nu(0) = \frac{2\xi_\nu}{\xi_\nu + \dfrac{\sqrt{3}}{2}}\left(a_\nu + \frac{b_\nu}{\sqrt{3}}\,\xi_\nu\right). \qquad \qquad (39)$$

In the adjacent continuous spectrum $s_\nu = 0$, and $n_\nu = 1 = \xi_\nu$. Denoting the net flux here by $\pi G_\nu(0)$ we have from (29)

$$G_\nu(0) = \frac{2}{1 + \dfrac{\sqrt{3}}{2}}\left(a_\nu + \frac{b_\nu}{\sqrt{3}}\right). \qquad \qquad (40)$$

Accordingly the intensity-ratio r_ν is given by

$$r_\nu = \frac{F_\nu(0)}{G_\nu(0)} = \xi_\nu\,\frac{1 + \dfrac{\sqrt{3}}{2}}{\xi_\nu + \dfrac{\sqrt{3}}{2}}\,\frac{1 + c_\nu\xi_\nu/\sqrt{3}}{1 + c_\nu/\sqrt{3}}, \qquad \qquad (41)$$

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* 'M.N.R.A.S.,' vol. 87, p. 697 (1927). The proof given in the text is, however, more general.

where

$$c_\nu = \frac{b_\nu}{a_\nu} = \frac{h\nu}{k\mathrm{T}_0}\frac{1}{\mathrm{T}_0}\left(\frac{d\mathrm{T}}{d\tau}\right)_0, = \frac{3}{8}\frac{h\nu}{k\mathrm{T}_0}, \qquad (42)$$

approximately, by (36A).

We now return temporarily to SCHUSTER's problem and ask what is the amount of matter per cm.^{2} which, scattering monochromatically with coefficient s_ν and otherwise transparent, would require to be placed in front of a surface radiating with intensity $\mathrm{G}_\nu(0)$ to give the same intensity-ratio r_ν as is given by (41). The answer is that an amount $\int \rho\, dh$ must be so placed, where

$$r_\nu = \frac{1}{1 + \frac{3}{4}\int s_\nu \rho\, dh} \qquad (43)$$

by (10). But

$$\frac{s_\nu}{\kappa} = \frac{1}{\xi_\nu^{\,2}} - 1$$

and

$$\kappa \int \rho\, dh$$

is the optical thickness τ of this column measured in *general* absorption (not selective scattering). Hence from (43)

$$r_\nu = \frac{\xi_\nu^{\,2}}{\xi_\nu^{\,2} + \frac{3}{4}\tau(1 - \xi_\nu^{\,2})}. \qquad (44)$$

In this formula τ is the optical thickness of the column in the real atmosphere which if placed in front of a photospheric radiating surface and deprived of all opacity except the power of scattering monochromatically, would produce an intensity-ratio r_ν at frequency ν equal to that in the actual stellar atmosphere. Thus τ is the optical thickness, or opacity, or fogginess, which the atmosphere would have to lose in order that the atoms, down to this level, when now fully viewed against a background, would produce the proper line-contour. Actually the atoms below τ are having some effect in causing conditions leading to the formation of the actual absorption line, but the atoms down to τ, "fully viewed," are equivalent to the atoms down to $\tau = \infty$, obscured at all depths as they are by general opacity.

So far as I can see, this is what is meant in the literature of the subject by speaking of a line being formed by the atoms down to a particular optical depth.

For equations (41) and (44) to give the same line contour, τ must be a function of ξ_ν. It is more convenient, however, to have τ as a function of r_ν, the observed intensity-ratio at ν. Eliminating ξ_ν between (41) and (44) we find a quadratic for τ. A simple approximation to the solution is obtained by writing

$$d_\nu = \frac{1 - \frac{1}{2}c_\nu}{1 + c_\nu/\sqrt{3}} = \frac{1 - \frac{3}{16}\frac{h\nu}{k\mathrm{T}_0}}{1 + \frac{\sqrt{3}}{8}\frac{h\nu}{k\mathrm{T}_0}}, \qquad (45)$$

and noting that d_ν is in practice small compared with unity.* We then find approximately

$$\tau = \tfrac{4}{3}\frac{r_\nu}{r_\nu + 1}\left[1 - \frac{2d_\nu}{(r_\nu + 1)(r_\nu + \frac{1}{2}\sqrt{3})}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

In applications it seems sufficient to take†

$$\tau = \tfrac{4}{3}\frac{r_\nu}{r_\nu + 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

The inner consistency of this with our solution to SCHWARZSCHILD's problem may be noted. In between the absorption lines we are viewing the equivalent photospheric surface. Here $r_\nu = 1$, and accordingly by (47) $\tau = \frac{2}{3}$. The temperature at this depth, deduced from $T^4 = \frac{1}{2}T_1^4(1 + \frac{3}{2}\tau)$ is $T = T_1$. Thus the equivalent photospheric radiating surface between the lines has to be placed at a depth where the true temperature is equal to the effective temperature—a most satisfactory result. More generally, the photosphere for intensity-ratio r_ν has to be placed at a depth where the temperature is T, given by

$$T^4 = T_1^4\frac{3r_\nu + 1}{2r_\nu + 2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

Our formula for the depth at which to place the "photosphere" has been obtained on a special assumption, namely the constancy of κ_ν/s_ν. But it may evidently be taken to give the right order of magnitude in general—τ varies with r_ν in just the way to be expected. We shall therefore use our formula in cases where our original assumptions do not necessarily hold. We shall make the assumption—and we argue that it is a fair assumption—that if we observe a point on an absorption-line contour where the intensity-ratio is r_ν, then study of the atoms giving rise to this absorption line may be confined to those contained in the uppermost $\frac{4}{3}r_\nu/(r_\nu + 1)$ of optical thickness. This assumption is only to be retained until more thorough mathematical analysis determines absorption-line contours under more general conditions. The advantage of the assumption is that it is easier to deal mathematically with a column (of finite length) of fully-viewed atoms than an infinite column of partially-viewed atoms lost at great depths in the general fog.

II.—*Determination of Photospheric Opacity.*

8. *Behaviour of Line-width from Star to Star.*—Let us now consider the same absorption line in a number of stellar spectra. Select in each case the point of the line-contour where the intensity-ratio r has some previously assigned value, say $r = \frac{1}{3}$. Determine the corresponding frequency ν. Then $|\nu - \nu_0|$, where ν_0 is the line-centre, measures the half-width of the line associated with this value of r. In general, $|\nu - \nu_0|$ will be

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* Its mean value through the spectrum is about zero.

† This is a revision of a formula $\tau = r/(r + 1)$, of inferior accuracy, given previously, 'M.N.R.A.S.,' vol. 89, p. 3 (1928).

different for the different spectra. But by SCHUSTER's formula the selective optical thickness σ is the same for all, since r is the same. Hence by (11), if N increases, α_ν must decrease (since $N\alpha_\nu$ remains constant) and hence $|\nu - \nu_0|$ must increase. Thus increasing N down to given τ is associated with increasing line-width for given r. Consequently study of line-widths for given r provides information about the behaviour of N down to given τ. In particular the line will be widest, for given r, when N is a maximum, for given τ.

Accordingly we must now show how to calculate the number of atoms in a given state of excitation and ionization down to given τ. SAHA showed* that the degree of ionization was calculable in terms of the temperature T and partial pressure of free electrons P. We know the range of temperature down to given τ by the solution to SCHWARZSCHILD's problem. We do not yet know how to calculate P.

9. *Integrated Ionization Formulæ.*—It is a comparatively simple matter to calculate the number of ionized or excited atoms in a column given the value of P at its base. The chief point requiring attention is that since P varies through the column from 0 at its upper end to say P_0 at its base, therefore the degree of ionization varies from point to point in the column. The variation of P along the column must be found from the mechanical equilibrium of the column. This depends on the value of g, which thus makes its first appearance in our analysis. The method of calculation† is best shown by a simple example.

Let x denote the degree of ionization of an element present to the extent of a proportion ε by mass and surrounded by atoms of the same mean ionization potential. Let χ_1 be the ionisation potential, K_1 the corresponding equilibrium constant calculated from the formula

$$K_1 = \frac{2\,(2\pi m e)^{3/2}\,(k\mathrm{T})^{5/2}\,e^{-\chi_1/k\mathrm{T}}\,q_2}{q_1 h^3} \quad \ldots \ldots \ldots \ldots \quad (49)$$

in a well-known notation. Then the degree of ionization x is given by

$$\frac{x}{1-x}\,\mathrm{P} = K_1 \quad \ldots \ldots \ldots \ldots \quad (50)$$

and the number of neutral and ionized atoms, N_0 and N_1, down to level $P = P_0$ are given by

$$N_0 = \frac{\varepsilon}{m}\int^{h_0}(1-x)\,\rho\,dh \quad \ldots \ldots \ldots \ldots \quad (51)$$

$$N_1 = \frac{\varepsilon}{m}\int^{h_0}x\,\rho\,dh, \quad \ldots \ldots \ldots \ldots \quad (52)$$

m being the atomic mass. Now by the equation of mechanical equilibrium the total pressure p is determined by

$$\frac{dp}{dh} = g\rho, \quad \ldots \ldots \ldots \ldots \quad (53)$$

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* 'Phil. Mag.,' vol. 40, p. 472 (1920).

† 'M.N.R.A.S.,' vol. 89, p. 17 (1928).

where g is the value of surface gravity, modified if necessary so as to take account of radiation pressure. Since the ionization of the surrounding atoms is the same as that of the atoms in question, the pressure p is connected with the partial electron pressure P by the relation

$$\frac{\mathrm{P}}{p} = \frac{x}{1+x}. \qquad (54)$$

Hence by (50), (53) and (54), (51) and (52) become

$$\mathrm{N}_0 = \frac{\varepsilon}{mg}\int_0^{\mathrm{P}_0} \frac{\mathrm{P}}{\mathrm{P}+\mathrm{K}_1}\, d\left[\frac{\mathrm{P}(\mathrm{P}+2\mathrm{K}_1)}{\mathrm{K}_1}\right] = \frac{\varepsilon}{mg}\frac{\mathrm{P}_0{}^2}{\mathrm{K}_1}, \qquad (55)$$

$$\mathrm{N}_1 = \frac{\varepsilon}{mg}\int_0^{\mathrm{P}_0} \frac{\mathrm{K}_1}{\mathrm{P}+\mathrm{K}_1}\, d\left[\frac{\mathrm{P}(\mathrm{P}+2\mathrm{K}_1)}{\mathrm{K}_1}\right] = \frac{\varepsilon}{mg}\,2\mathrm{P}_0, \qquad (56)$$

on taking a mean value for T so that K_1 may be treated as a constant.

More complicated cases may be dealt with similarly. For example when the element is surrounded by an excess of atoms just once-ionized, as may be taken to occur for stars of type G_0 and earlier, (55) and (56) become replaced by

$$\mathrm{N}_0 = \frac{2\varepsilon}{mg}\left[\mathrm{P}_0 - \mathrm{K}_1 \log\left(1 + \frac{\mathrm{P}_0}{\mathrm{K}_1}\right)\right] \qquad (55')$$

$$\mathrm{N}_1 = \frac{2\varepsilon}{mg}\,\mathrm{K}_1 \log\left(1 + \frac{\mathrm{P}_0}{\mathrm{K}_1}\right), \qquad (56')$$

which approximate to (55) and (56) when K_1 is large. The main point to note about these formulæ is that variations of electron-pressure in the column have been accurately allowed for and that the resulting formulæ are of the type

$$\mathrm{N}_0, \mathrm{N}_1 = \frac{\varepsilon}{mg} f(\mathrm{P}_0, \mathrm{T}). \qquad (57)$$

10. *Electron-pressure and Optical Depth.*—It remains to calculate the electron-pressure P_0 at the optical depth τ. This can only be done when the absorption coefficient of the photospheric material is known. We require the physical law determining the opacity of unit mass of the photospheric material in terms of its temperature and pressure. This is at present unknown from pure physics, and we seek to use our analysis of stellar atmospheres to determine it from observation. Clearly the greater is the absorption coefficient κ, the smaller the amount of mass required to establish a given total opacity τ, and the smaller is the pressure at the base of the column. The order of magnitude of this pressure, and the complete structure of the photospheric layers as a function of depth measured in kilometres, depend on the absolute value of κ. A knowledge of the value of κ is the key which would unlock the whole position.

A simple assumption would be to take κ constant, say $\bar{\kappa}$, independent of the physical state of the material. But in the interior of a star, EDDINGTON* has shown that opacity is due to the photo-electric effect. If we make the same assumption for the photospheric layers, we are led to the formula

$$\kappa = \frac{\alpha P}{T^{9/2}}, \quad \ldots \ldots \ldots \ldots \quad (58)$$

where α is roughly independent of P and T but depends on the chemical composition. It is interesting to work out the results of both assumptions.

If $\kappa = \bar{\kappa}$, we have†

$$\tau = \int^{h_0} \bar{\kappa} \rho \, dh = \frac{\bar{\kappa}}{g} p_0 = \frac{\bar{\kappa}}{g} P_0 \left(2 + \frac{P_0}{K_1}\right). \quad \ldots \ldots \ldots \quad (59)$$

on using (53), (50) and (54) as before.

If (58) holds, we have

$$\tau = \int^{h_0} \frac{\alpha P}{T^{9/2}} \rho \, dh = \frac{\alpha}{g T^{9/2}} P_0^2 \left(1 + \tfrac{2}{3} \frac{P_0}{K_1}\right) \quad \ldots \ldots \ldots \quad (59')$$

on taking a mean value for T.

We have now completed the determination of the members of relevant atoms concerned in producing an absorption line of given r. Provided $\bar{\kappa}$ or α were known, (59) or (59′) determines the base pressure P_0 given τ and T, and then (55) and (56) or similar formulæ determine N_0 and N_1.

10A. "*Mass above the Photosphere*" *and Photospheric Opacity.*—Before we show how to determine the absolute value of the photospheric absorption coefficient we will enquire whether it is possible to distinguish observationally between $\kappa =$ constant and $\kappa = \alpha P/T^{9/2}$. Let us calculate the total mass μ down to level τ on the two assumptions.

In each case $\mu = p_0/g$. Hence by (59), when $\kappa =$ constant $= \bar{\kappa}$,

$$\mu = \tau/\bar{\kappa}. \quad \ldots \ldots \ldots \ldots \ldots \quad (60)$$

When $\kappa = \alpha P/T^{9/2}$, by (59′), (50) and (54)

$$\mu = \tau \frac{T^{9/2}}{\alpha} \frac{2 + P_0/K_1}{P_0 (1 + \frac{2}{3} P_0/K_1)}. \quad \ldots \ldots \ldots \quad (60')$$

We see that when $\kappa =$ constant, μ is independent of g. But when $\kappa = \alpha P/T^{9/2}$, by (60′) μ increases as P_0 decreases; by (59′), P_0 decreases as g decreases; and thus μ increases as g decreases. The fundamental difference between $\kappa =$ constant and $\kappa = \alpha P/T^{9/2}$ is that in the former case the total "mass above the photosphere" is independent of g and therefore the same for giants and dwarfs, but in the latter case the total "mass above the photosphere" increases, at constant T, as we pass from dwarfs to giants.

The observed fact is that all lines are strengthened from dwarfs to giants. We cannot immediately hail this as evidence in favour of the law $\kappa \propto$ P, for the decrease of P_0

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* 'M.N.R.A.S.,' vol. 83, p. 32 (1922) and vol. 84, p. 104 (1924).

† Formulæ (59) and (59′) are calculated for an element surrounded by atoms of the same mean I.P.

from dwarfs to giants causes increased ionization, which affects neutral and ionized atoms differently. The effect is however calculable, and it is found* that the law $\kappa = \alpha P/T^{9/2}$ in every case predicts the right change of intensity between dwarfs and giants, whilst in many cases $\kappa =$ constant predicts a change in the opposite direction. Full details have been given elsewhere.

The main point of physical interest is that the reduced pressure gradients in giants as compared with dwarfs do not of themselves lead to an increased " mass above the photosphere " ; to secure the latter, the material must be *intrinsically*† more transparent under giant conditions. This suggests that the coefficient of opacity must decrease with decreasing pressure. We do not establish the detailed law $\kappa = \alpha P/T^{9/2}$, but we obtain confirmation of its main feature, the proportionality with P.

11. *The " Method of Maxima " for Determination of Photospheric Opacity.*—Having settled this point, we will now show how the absolute value of the constant α may be derived from stellar data. The principle of the method‡—conveniently described as the " method of maxima "—is as follows. Maxima of certain absorption lines in the stellar sequence of spectra are well known. It is found that such lines originate from either normal states of ionized atoms, or excited states of atoms either neutral or ionized. For simplicity we will confine the argument to the excited state of a neutral atom. The maximum results from the interplay of ionization and excitation with change of temperature. Increasing temperature decreases the proportion of neutral atoms, but increases the fraction of these which are excited, and the result is a rise to a maximum followed by a steady decline. At low temperatures the fraction of atoms neutral is large, but the fraction of these excited is small ; at high temperatures the fraction excited is larger but the proportion of neutral is small. In between there is a maximum.

It was shown by FOWLER and MILNE§ that the maximum *concentration* of excited neutral atoms reaches a maximum at a temperature determined by P, the electron-pressure. Conversely, from an observed temperature of maximum, ($T_{\text{max.}}$), P could be deduced. It was supposed that the value P so obtained was some kind of average through the stellar atmosphere, though it was quite uncertain to what level it referred.

The formulæ now developed permit this uncertainty to be removed. In accordance with the ideas of STEWART and UNSÖLD, as here extended, what determines the maximum width of a line for given intensity-ratio r is not the maximum *concentration* of excited atoms but the maximum *number* of excited atoms down to the associated τ.

If $A_0^{(s)}$ is the excitation factor for neutral atoms (a function of the temperature T and the excitation potential), then the number of excited neutral atoms in the column is $N_0^{(s)}$, given by

$$N_0^{(s)} = A_0^{(s)} N_0,$$

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* ' M.N.R.A.S.,' vol. 89, p. 157 (1928).

† *i.e.*, more transparent per unit mass per cm.^{2}.

‡ ' M.N.R.A.S.,' vol. 89, p. 17 (1928).

§ ' M.N.R.A.S.,' vol. 83, p. 403 (1923), and vol. 84, p. 499 (1924).

where N_0 is to be calculated from a formula of the type (55) or (57), P_0 being determined by a formula of the type (59′) and τ being determined from r, obtained from the line-contour. We have thus an expression

$$N_0^{(s)} = \frac{\varepsilon}{mg}\,\phi\,(T, P_0), \qquad (61)$$

and we have to make this a maximum for variation of T, P_0 being a function of T varying according to a relation of the type

$$\tau = \frac{\alpha}{g}\,\phi_1\,(T, P_0). \qquad (62)$$

The ordinary process of differentiating for a maximum leads to an equation

$$\frac{\partial\phi}{\partial T}\frac{\partial\phi_1}{\partial P_0} - \frac{\partial\phi}{\partial P_0}\frac{\partial\phi_1}{\partial T} = 0, \qquad (63)$$

valid at $T = T_{\text{max.}}$ This equation, which is explicitly independent of ε, α and g, can be solved for P_0 if $T_{\text{max.}}$ is known from observation, and thus the partial electron pressure at the base of the column is determined. Introduction of this value of P_0 into (62), τ being already known from r, evaluates α. Thus the constant in the expression for the coefficient of opacity is derived from observation. Once α is determined, the coefficient of opacity may be calculated at any desired level in the atmosphere.

The explicit form of (63) has been given elsewhere* for a variety of cases. The solution is found to be expressible in all cases in the form

$$P_0 = [\theta K_1]_{T = T_{\text{max.}}} \qquad (64)$$

where θ depends on the excitation and ionization potentials concerned but varies only very slowly with $T_{\text{max.}}$ The applicability of the method is closely connected with the fact that (64) is explicitly independent of ε, the proportion by mass of the element considered. It has merely been necessary to assume ε constant in differentiating.

The physical reason why observation of a maximum determines the *general* opacity coefficient is worth considering. The column of atoms has to contain a given total opacity, τ, and at the same time to contain a maximum number of excited atoms, in order that a line shall be widest for given intensity-ratio. The temperature of maximum depends on the partial electron-pressure, and this can be thus found from an observed $T_{\text{max.}}$ We thus know two things about the column—its total opacity and the electron pressure at its base. From the latter we can infer from consideration of mechanical equilibrium what is the total mass of the column. Dividing the total mass into the total opacity we get the opacity per gram, which is precisely the coefficient of opacity. Using once again the value of P_0, we find the constant α. Free electrons play two distinct parts—they determine the ionization and they determine the opacity. But the method does not depend on this two-fold appearance of free electrons. Whatever the source of

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* ‘M.N.R.A.S.,’ vol. 89, p. 17 (1928).

opacity, the method gives the opacity coefficient. But the law $\kappa = \alpha P/T^{9/2}$ makes the details of the determination both simple and interesting.

The necessity for an appeal to the condition of mechanical equilibrium shows that g must be known. Consequently it is necessary to determine a maximum for stars of constant g, or else to introduce the actual variation of g in performing differentiations.

12. *Numerical Evaluation of the Absorption Coefficient.*—We now apply the whole of the preceding theory to actual observations. It is recorded by MENZEL and by Miss PAYNE* that the Zn triplet $1\,{}^3P$—$1\,{}^3S$ $\lambda\lambda$ 4810, 4722, (4680) has a maximum in stars of type G_0. Amongst many recorded maxima we choose this as a fundamental one, because we have two stars of type G_0, namely the Sun and Capella, for which the values of g and T are fairly accurately known. We shall take T to be equal to T_1, the effective temperature. It would be easy to determine the true mean temperature $\overline{T}$ of the layers down to τ, but it is not certain that the ionization would be accurately calculable from $\overline{T}$, since the material is traversed by radiation of temperature T_1. Whilst recognising that the best temperature to take is one lying between $\overline{T}$ and T_1, we shall adopt $T = T_1$ for simplicity.† (The point is discussed further below in relation to the BALMER maximum.) We adopt $\tau = \frac{1}{3}$, corresponding to $r = \frac{1}{3}$. Actually line-contour determinations of these lines are not available but $\tau = \frac{1}{3}$ cannot be far wrong. The formulæ employed are those of "Problem III" of a previous paper,‡ namely those appropriate to "atoms becoming ionized in the presence of an excess of atoms already once ionized"; for this problem $\theta = P_0/K_1$ is the root of the equation

$$\frac{\log(1+\theta) - \theta/(1+\theta)}{\theta - \log(1+\theta)} - \frac{\frac{9}{4}kT}{\chi_1 + \frac{5}{2}kT}\,\frac{\theta^2/(1+\theta)}{\theta - \log(1+\theta)} = \frac{\chi_1 - \chi_1^{(s)}}{\chi_1 + \frac{5}{2}kT},$$

where T is to be put equal to $T_{max.}$. The results of the calculations are as follows.§

TABLE I.—Maximum of Zn lines $1\,{}^3P$—$1\,{}^3S$, $\lambda\lambda$ 4810, 4722 (4680), Type G_0.
$\chi_1 = 9 \cdot 34$ volts, $\chi_1^{(s)} = 5 \cdot 33$ volts, $q_0 = 1$, $q_1 = 2$.

Star.	Sun.	Capella (*b*).
Temperature $T_1 = T_{max.}$	5,740°	5,200°
Surface gravity $= g$	$2 \cdot 74 \times 10^4$	$6 \cdot 08 \times 10^2$ cm. sec.$^{-2}$
Equilibrium constant $= K_1$	20·8	2·20
θ	1·48	1·56
Degree of ionization $x_0 = 1/(1+\theta)$	0·403	0·390
Electron-pressure at base $P_0 = \theta K_1$	30·8	3·43 dyne cm.$^{-2}$
Absorption coefficient $\kappa_0 = \tau g/P_0$	297	59·0 gram.$^{-1}$ cm.$^{-2}$
$\alpha \times 10^{-18} = \kappa_0\,(T/10^4)^{9/2}/P_0$	0·794	0·911

[The suffix 0 refers throughout to the level $\tau = \frac{1}{3}$.]

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* MENZEL, 'Harvard Circ.,' p. 258 (1924); PAYNE, 'Stellar Atmospheres,' p. 130 (1925).

† *Cf.* GERASIMOVIČ, 'Proc. Nat. Acad. Sci.,' vol. 13, p. 180 (1927).

‡ 'M.N.R.A.S.,' vol. 89, p. 17 (1928).

§ In 'M.N.R.A.S.,' vol. 89, p. 35 (1928), I adopted $\chi_1 = 7 \cdot 5$ volts, whilst pointing out that this was probably too low. The opacities now calculated are considerably higher, but seem much more trustworthy.

If the data and theory were quite accurate the two values of α should agree. The two values are $\alpha = 0{\cdot}79 \times 10^{18}$ and $\alpha = 0{\cdot}91 \times 10^{18}$. Considering the differences in T and g between the two stars, the agreement seems satisfactory.

It should be borne in mind that the method can only be expected to give an accurate value of α if either χ_1 for given $T_{\text{max.}}$ or $T_{\text{max.}}$ for given χ_1 are known very accurately. For $(K_1)_{T\,\text{max.}}$ is chiefly determined by the factor $e^{-\chi_1/kT\,\text{max.}}$, which is sensitive to small errors in χ_1 or $T_{\text{max.}}$; P_0 is determined by K_1; and lastly the formula

$$\tau = \frac{\alpha}{g T^{9/2}} P_0^2$$

shows that α is determined by the square of P_0.

In what follows, we shall adopt the value (the mean of the two determinations)

$$\alpha = 0{\cdot}85 \times 10^{18},$$

so that the photospheric absorption coefficient is given by

$$\kappa = 0{\cdot}85 \frac{P}{(T/10^4)^{9/2}}.$$

13. *Determination from the Maximum of the* BALMER *Lines.*—Miss PAYNE and Miss WILLIAMS have recently* made an accurate determination of the maximum of the BALMER lines in A-type stars. They find that for $r = 0{\cdot}48$ it occurs at type A_0, for $r = 0{\cdot}96$ at types A3–A5. (We will later discuss the variation of maximum with r.) There is some uncertainty as to the proper temperature-scale for the A-type stars. We shall average the observations by assuming that at $r = 0{\cdot}7$, the maximum occurs at A2, $T_1 = 10{,}000°$. For $r = 0{\cdot}7$, $\tau = 0{\cdot}55$. For g we adopt the value for Sirius deduced from the mass and luminosity recorded by EDDINGTON. (It is not suggested that Sirius is of type A2; we merely use Sirius to provide a g-value.) The formulæ employed are those for "Problem III" as before (material on the whole once ionized). The results of the calculation are given in Table II.

The accuracy of this calculation is capable of considerable improvement when better values of $T_{\text{max.}}$ and g are available. We note however that α comes out of the same order of magnitude as for the photospheres of the Sun and Capella—about twice as large. The agreement is on the whole as close as could be expected.

We now consider the variation in the position of maximum as we move from the centre of the line to the wings. The observations of Miss PAYNE and Miss WILLIAMS show that as we move out to the wings (r and τ increasing) $T_{\text{max.}}$ decreases. I originally predicted† that $T_{\text{max.}}$ should increase with increasing r. Increasing r or τ means seeing to increased depth, *i.e.*, increased pressure; increased pressure retards ionization and

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* 'M.N.R.A.S.,' vol. 89, p. 526 (1929).

† 'M.N.R.A.S.,' vol. 89, p. 17 (1928).

therefore would appear to require increased temperature to compensate. This was the crude argument. But I omitted to notice that increased depth τ means not only increased pressure but also increased temperature. Miss PAYNE has suggested to me in correspondence that this might cause an effect of the opposite sign. The argument is that the decreased ionization consequent on increased pressure may be more than compensated by the increased ionization consequent on increased temperature, so that on balance the temperature of maximum may be decreased and not increased, as observed. Mathematical analysis confirms Miss PAYNE's conjecture. We have in fact an interesting application of the solution to SCHWARZSCHILD's problem.

TABLE II.—Maximum of BALMER Lines for $r = 0 \cdot 7$, taken as Type A2.

$\chi_1 = 13 \cdot 54$ volts, $\chi_1^{(s)} = 3 \cdot 385$ volts, $q_1 = 1$, $q_0 = 2$.

Temperature $= T_1 = (T)_{max.}$	10,000°
Surface gravity $= g$	$2 \cdot 64 \times 10^4$
Equilibrium constant $= K_1$	$4 \cdot 82 \times 10^2$
θ*	$0 \cdot 203$
Degree of ionization $= x_0 = 1/(1 + \theta)$	$0 \cdot 830$
Electron-pressure at base $= P_0 = \theta K_1$	0978×10^2 dyne cm.$^{-2}$
Absorption coefficient $= \kappa_0 = \tau g/P_0$	148 gram^{-1} cm.$^{-2}$
$\alpha \times 10^{-18} = \kappa_0 (T/10^4)^{9/2}/P_0$	$1 \cdot 51$

[*] Formerly I estimated $\theta = 1$. Actual solution of the equation for θ shows it to be much less than unity. In my previous analysis of the BALMER maximum I found $\kappa_0 = 7$. In the present analysis I take g 2·5 times larger, P_0 comes out 5 times larger owing to $\theta = 0 \cdot 2$, and I take τ about $1\frac{2}{3}$ times larger, giving a total factor 21 times larger. Hence $\kappa_0 = 148$.

The temperature of maximum is determined by the equation $P_0 = [\theta K_1]_{T_{max.}}$ where θ varies only slowly with $T_{max.}$ and K_1 varies principally through its exponential factor containing $T_{max.}$. What we actually observe is not, however, $T_{max.}$ but $(T_1)_{max.}$, where

$$(T_{max.})^4 = \tfrac{1}{2} (T_1)^4_{max.} (1 + \tfrac{3}{2} \tau),$$

and τ is related to P_0 by an equation already given. Ignoring slowly varying factors, we have in principle

$$P_0 \propto e^{-\chi_1/k (T_1)_{max.} (1+\frac{3}{2}\tau)^{\frac{1}{4}}},$$

$$\tau \propto P_0^2,$$

or

$$\tau \propto e^{-2\chi_1/k (T_1)_{max.} (1+\frac{3}{2}\tau)^{\frac{1}{4}}}.$$

The crude argument originally put forward by myself ignores the occurrence of τ on the right-hand side of this equation, for it ignores the variation of true temperature with depth. In that case $(T_1)_{max.}$ would increase with increasing τ. But differentiating the proportionality as it stands, we have

$$\frac{d\tau}{\tau} = \frac{2\chi_1}{k (T_1)_{max.} (1 + \frac{3}{2} \tau)^{\frac{1}{4}}} \left[\frac{d (T_1)_{max.}}{(T_1)_{max.}} + \frac{\frac{3}{8} d\tau}{1 + \frac{3}{2} \tau}\right].$$

This shows that $d\,(T_1)_{\max.}/d\tau$ will be negative provided

$$\tfrac{1}{2} < \frac{2\chi_1}{kT} \cdot \frac{\frac{3}{8}}{1 + \frac{3}{2}\tau}$$

or

$$\frac{\chi_1}{kT} > \frac{\frac{4}{3}(1 + \frac{3}{2}\tau)}{\tau}. \qquad (65)$$

For the Balmer maximum at 10,000°, χ_1/kT certainly exceeds 15·7, since T is less than T_1. For $r = 0·30$ (the observed central residual intensity) τ is $\frac{4}{5}(0·3)/(1·3) = 0·308$ and $\frac{4}{3}(1 + \frac{3}{2}\tau)/\tau = 6·3$. Thus the inequality is satisfied, and $d\,(T_1)_{\max.}/d\tau$ is negative, as observed. Increased temperature with increased depth causes more ionization than is necessary to compensate for the increased pressure, and the maximum occurs at a lower effective temperature.

The foregoing amounts to a second approximation to the theory. In the first approximation we took a mean value for T through the layer contributing the absorption line, and put it equal to T_1. In the second approximation we take into account the probable variation of T through the layer as given by the solution to Schwarzschild's problem. For simplicity we have taken the actual temperature at the base of the layer as determining the ionisation, in this second approximation.

III.—*Applications of the Absolute Value of the Photospheric Absorption Coefficient.*

14. *Application to Ionization of Calcium at Low Temperatures.*—Once α is determined, the number of atoms per cm.^{2} down to given τ may be calculated for any element in terms of an assumed composition by mass. We illustrate by calculating the numbers of Ca and Ca$^+$ atoms (N_0 and N_1) down to $\tau = \frac{1}{3}$, in atmospheres of various T_1 and g, assuming calcium forms 5 per cent. of the mass of the atmosphere. The formulæ used are (55), (56) and (59′) with $\alpha = 0·85 \times 10^{18}$, $\tau = \frac{1}{3}$, $\varepsilon = 0·05$, $m = 40 \times 1·66 \times 10^{24}$, $\chi_1 = 6·08$. The calculations are confined to low temperatures (3500°, 4000°, 4500°) so that second-stage ionization of calcium may be ignored. The following tables (Tables III, IV and V) show the calculated values of P_0, $\log_{10} N_0$, $\log_{10} N_1$, $\log_{10} (N_1/N_0)$ and $\log_{10} (N_0 + N_1)$ for various values of g, for three different temperatures. The calculations have been performed by choosing the values of P_0 in the first instance, so as to avoid the solution of a cubic equation, but in each table g should be regarded as the actual independent variable.

The results are plotted as curves in figs. 1, 2. Fig. 1 shows $\log P_0$ as a function of g at temperatures 3500°, 4000°, 4500°. It will be seen that the variation of P_0 with g is much more pronounced than the variation with T. Fig. 2 shows $\log N_0$, $\log N_1$, $\log (N_1/N_1)$ and $\log (N_0 + N_1)$ as functions of g for the three temperatures. These may be regarded as the predicted "absolute magnitude" effects at various temperatures for the relative intensities of λ 4227 (Ca) and λλ 3933, 3968 (Ca$^+$). It will be seen that N_0

TABLE III.—Ionization of Calcium in Stellar Atmospheres.

N_0 = Number of Ca atoms down to $\tau = \frac{1}{3}$ ⎫ Assumed composition, 5 per cent. of
N_1 = Number of Ca^+ atoms down to $\tau = \frac{1}{3}$ ⎭ calcium by mass ($\varepsilon = 0{\cdot}05$).
P_0 = electron pressure at $\tau = \frac{1}{3}$.

T = 3500°

$\log_{10} g$ (g in cm. sec.$^{-2}$)	P_0 dynes cm.$^{-2}$	$\log_{10} N_0$	$\log_{10} N_1$	$\log_{10} (N_1/N_0)$	$\log_{10} (N_0 + N_1)$
$\bar{2}{\cdot}46$	10^{-2}	18·21	20·72	2·51	20·72
0·48	10^{-1}	18·19	19·70	1·51	19·71
1·94	5×10^{-1}	18·13	18·94	0·81	19·00
2·61	10°	18·06	18·57	0·51	18·69
4·34	$5 \times 10^{\circ}$	17·72	17·54	$\bar{1}{\cdot}82$	17·94
5·17	10	17·50	17·01	$\bar{1}{\cdot}51$	17·62
7·19	5×10	16·87	15·69	$\bar{2}{\cdot}82$	16·90

TABLE IV.—Ionization of Calcium in Stellar Atmospheres.

T = 4000°.

$\log_{10} g$	P_0 dynes cm.$^{-2}$	$\log_{10} N_0$	$\log_{10} N_1$	$\log_{10} (N_1/N_0)$	$\log_{10} (N_0 + N_1)$
$\bar{2}{\cdot}20$	10^{-2}	17·23	20·98	3·75	20·98
0·20	10^{-1}	17·23	19·98	2·75	19·98
2·21	10°	17·22	18·97	1·75	18·98
3·64	$5 \times 10^{\circ}$	17·18	18·23	1·05	18·27
4·29	10	17·13	17·89	0·76	17·96
5·93	5×10	16·89	16·94	0·05	17·22

TABLE V.—Ionization of Calcium in Stellar Atmospheres.

T = 4500°.

$\log_{10} g$	P_0 dynes cm.$^{-2}$	$\log_{10} N_0$	$\log_{10} N_1$	$\log_{10} (N_1/N_0)$	$\log_{10} (N_0 + N_1)$
$\bar{3}{\cdot}97$	10^{-2}	16·48	21·21	4·73	21·21
$\bar{1}{\cdot}97$	10^{-1}	16·48	20·21	3·73	20·21
1·37	5×10^{-1}	16·48	19·51	3·03	19·51
1·97	10°	16·47	19·21	2·74	19·21
3·37	$5 \times 10^{\circ}$	16·47	18·51	2·04	18·51
3·98	10	16·47	18·20	1·73	18·21
5·42	5×10	16·43	17·46	1·03	17·50
6·06	10^{2}	16·38	17·11	0·73	17·19
8·51	10^{3}	15·94	15·67	$\bar{1}{\cdot}73$	16·13

decreases slowly, N_1 quickly, with increasing g. The curve for log (N_1/N_1) is independent of the assumed value of ε. The other curves may be displaced bodily up or down to correspond to different values of ε. The curves for log $(N_0 + N_1)$ represent the total amount of calcium "above the photosphere." By adjustment of ε they could be taken to represent the total amount of any other element "above the photosphere." In particular, for an element only slightly ionized at low temperatures they represent the strength of the neutral lines as a function of g. Thus, for example, they give the predicted behaviour of hydrogen intensities at low temperatures as a function of g.

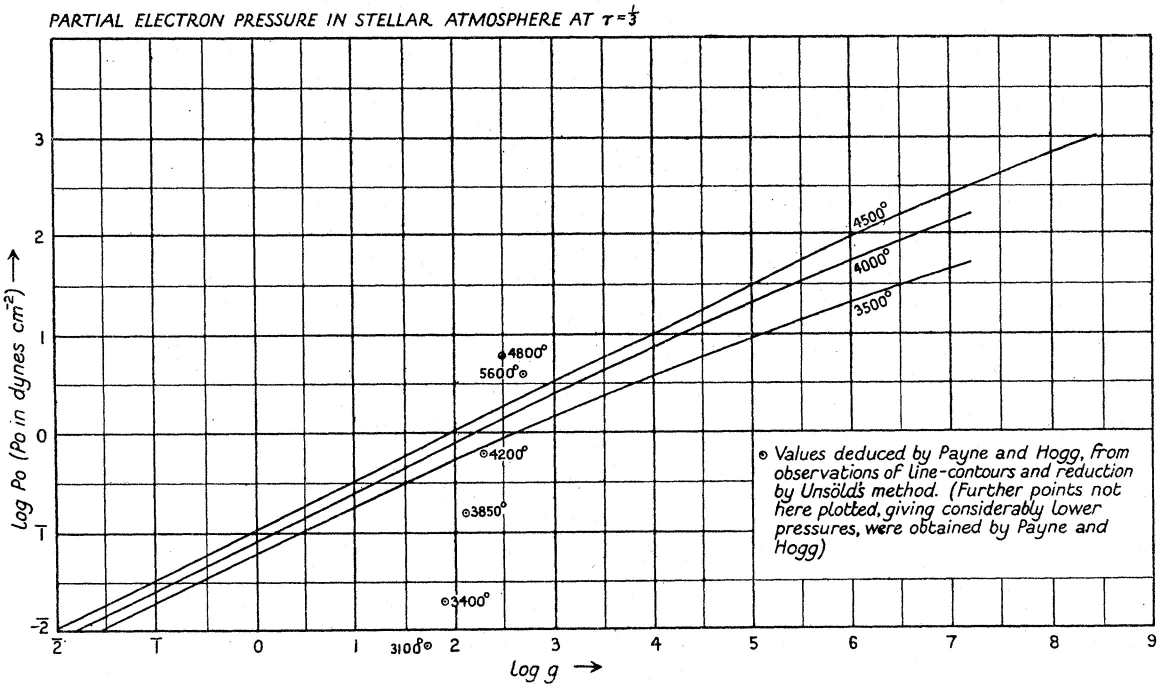
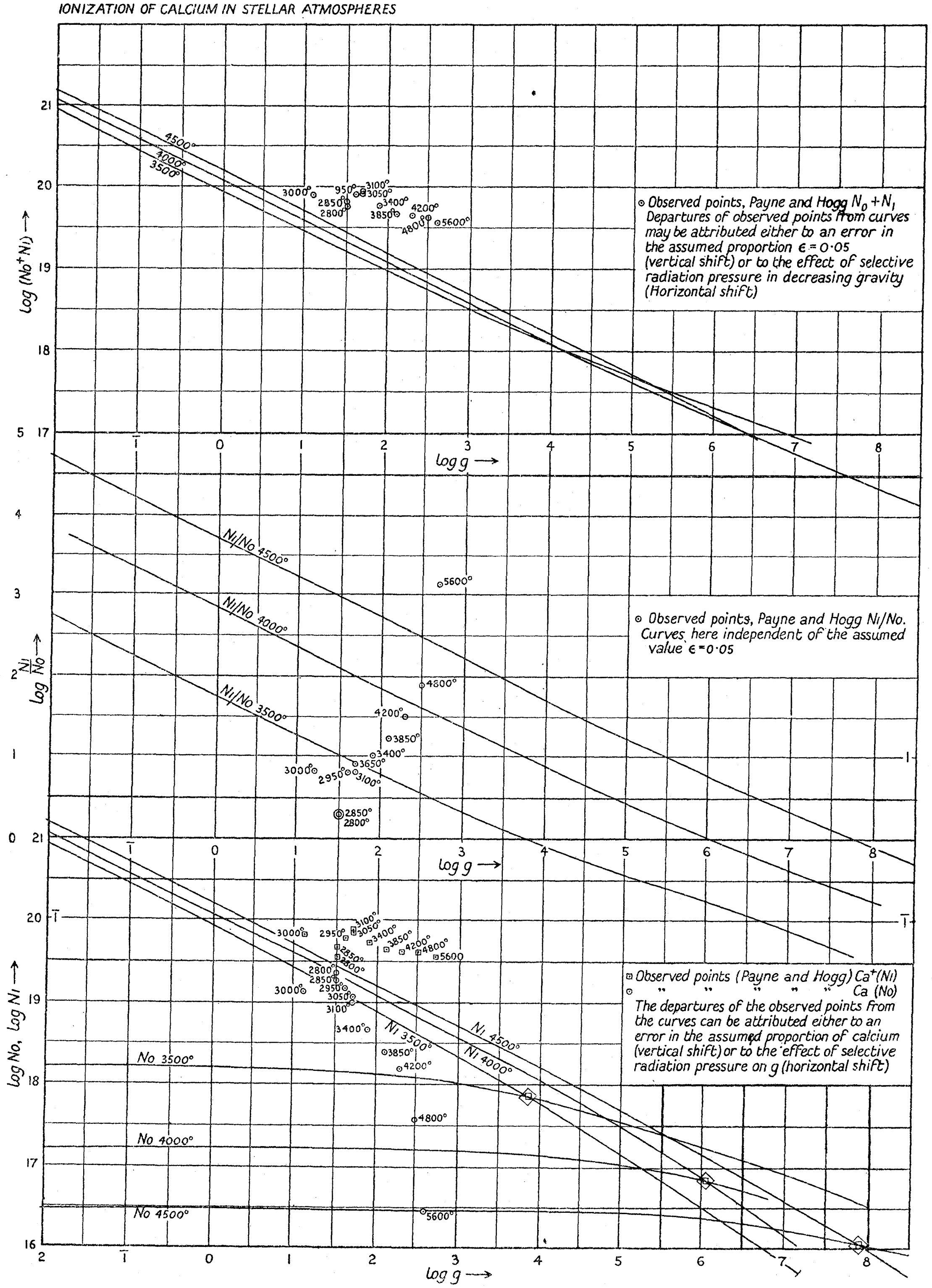


FIG. 1.—*Partial electron pressures* P_0 *in stellar atmospheres*—at low temperatures at depth $\tau = \frac{1}{3}$, calculated as a function of surface gravity g.

Note.—The observed points correspond to pressures $\overline{P}$ deduced by PAYNE and HOGG from the values of N_0 and N_1 for Ca, as obtained by UNSÖLD's method. They show a much more rapid decrease of pressure with gravity than shown by the calculated curves.

It should be remembered that the only astrophysical datum employed in calculating these curves is the value $\alpha = 0 \cdot 85 \times 10^{18}$, derived from the observed maximum of the Zn lines in G_0 type stars. No observed intensities of calcium lines have been introduced. It is therefore of great interest to compare these predicted effects with actual observations.

PAYNE and HOGG (*loc. cit.*) have applied UNSÖLD's method to the contours of H, K and λ 4227 in the cooler stars. Using UNSÖLD's formula for the atomic scattering coefficient—which has not been required at any point in our work—they have derived values of log N_0, log N_1, log $(N_0 + N_1)$ and log (N_1/N_0). From the latter they have



IONIZATION OF CALCIUM IN STELLAR ATMOSPHERES

⊙ Observed points, Payne and Hogg N_0+N_1 Departures of observed points from curves may be attributed either to an error in the assumed proportion ε = 0·05 (vertical shift) or to the effect of selective radiation pressure in decreasing gravity (Horizontal shift)

⊙ Observed points, Payne and Hogg Ni/No. Curves here independent of the assumed value ε = 0·05

⊡ Observed points (Payne and Hogg) Ca⁺(Ni)
⊙ " " " " " Ca (No)
The departures of the observed points from the curves can be attributed either to an error in the assumed proportion of calcium (vertical shift) or to the effect of selective radiation pressure on g (horizontal shift)

FIG. 2.—*Ionization of calcium in stellar atmospheres at low temperatures, as a function of surface gravity g.* —Assumed proportion of calcium 5 per cent. by mass ($\varepsilon = 0\cdot05$); assumed photospheric opacity, $\kappa = 0\cdot85\,P/(T/10^4)^{9/2}$ (P = electron-pressure in dynes cm^{-2}.; N_0 = number of neutral atoms down to $\tau = \frac{1}{3}$ per cent.; N_1 = number of ionized atoms down to $\tau = \frac{1}{3}$ per cent.

Note (1).—$N_0 + N_1$ is the "total number of atoms above the photosphere" (actually above $\tau = \frac{1}{3}$). It represents the total number of neutral atoms of *any* constituents ionizable only with difficulty, and thus gives, for example, the behaviour of BALMER lines with gravity at low temperatures.

Note (2).—The single datum from which the curves are calculated is the observed Zn maximum in G_0 type stars.

Note (3).—The observed points were all obtained by an entirely independent method, the line-contours method of UNSÖLD, assuming a formula for the atomic scattering coefficient. The fact that the calculated curves pass through the neighbourhoods of the observed points checks the general correctness of the aggregate of assumptions made and formulæ used.

determined $\log \overline{P}$. Their results, transcribed from Harvard Circular, p. 334 (1928) are as follows.

TABLE VI.—Observed Ionization of Calcium in Stellar Atmospheres. (PAYNE and HOGG.)

Spectral Class.	T_1	$\log g$	$\log N_0$	$\log N_1$	$\log (N_0+N_1)$.	$\log \overline{P}$
G_0	5600	2·7	16·44	19·55	19·56	+0·6
G_5	4800	2·5	17·56	19·61	19·61	+0·8
K_0	4200	2·3	18·18	19·61	19·63	−0·2
K_2	3850	2·1	18·39	19·63	19·66	−0·8
K_5	3400	1·9	18·66	19·72	19·76	−1·7
M_0	3100	1·7	19·00	19·90	19·95	−2·2
M_1	3050	1·7	19·05	19·87	19·93	−2·4
M_2	3000	1·1	19·11	19·82	19·90	−2·5
M_3	2950	1·6	19·16	19·78	19·90	−2·7
M_5	2850	1·5	19·26	19·67	19·81	−2·6
M_7	2800	1·5	19·35	19·55	19·76	−2·8

The whole of these observations (with the exception of some of the smaller $\overline{P}$'s) are plotted on the appropriate diagrams in figs. 1, 2. Actually the values of $\overline{P}$ should have been doubled (*i.e.*, 0·30 added to their logarithms) before being plotted, as PAYNE and HOGG calculate $\overline{P}$ from the formula

$$\frac{x_2}{x_1}\overline{P} = K_1,$$

whilst (55) and (56) give

$$\frac{N_2}{N_1}P_0 = 2K_1,$$

so that taking $x_2/x_1 = N_2/N_1$, $P_0 = 2\overline{P}$. Our curves give P_0, PAYNE and HOGG's points refer to $\overline{P}$.

The observed values of $\overline{P}$ decrease with decreasing g much more rapidly than predicted. This is an outstanding discrepancy. Apart from this the observed points cluster well round the calculated curves. This shows that $\varepsilon = 0·05$ is roughly correct for the calcium content. But it is undesirable to press this. The main conclusion is that the theory of opacity put forward in this lecture, and the determination of its absolute amount, is consistent with the determinations of content by UNSÖLD's method, based on entirely different data, both theoretical and observational.

Developments of the method at once suggest themselves. Once the value of ε is determined with some approach to accuracy for a few typical stars, it may presumably be used for other stars. A determination of N_1 or N_0 by UNSÖLD's method for a star of known T but unknown g can then be plotted on the appropriate curve and the corresponding value of g read off. We have thus a method of determining g-values for separate stars, independent of empirical calibration curves such as are used in the method of spectroscopic parallaxes. The method is applicable to any line for which line-contours can be determined. The ratio N_1/N_0 can be used, alternatively, instead of the separate values of N_1 and N_0, and this has certain advantages.

15. *Application to the* BALMER *Lines.*—Tables VII and VIII show the numbers of

neutral atoms of hydrogen in a stellar atmosphere down to $\tau = \frac{1}{3}$, and the numbers of excited atoms capable of absorbing BALMER lines, for various g and T, on the two laws (1) $\kappa = 0{\cdot}85\mathrm{P}/(\mathrm{T}/10^4)^{9/2}$, (2) κ = constant = 100. The first should represent the actual behaviour of BALMER lines in stellar atmospheres; the second is merely inserted for contrast. The values of log $\mathrm{N}_0^{(s)}$ (numbers of excited atoms) are plotted in figs. 3 and 4.

TABLE VII.—Hydrogen. Calculated numbers of atoms per cm.² down to $\tau = \frac{1}{3}$ capable of absorbing BALMER lines ($\mathrm{N}_0^{(s)}$) for various g and T, assuming 5 per cent. by mass of hydrogen, for $\kappa = 0{\cdot}85\ \mathrm{P}/(\mathrm{T}/10^4)^{9/2}$. ($\mathrm{N}_0$ = number of neutral atoms.)

T.	$g = 10$.		$g = 10^2$.		$g = 10^3$.		$g = 10^4$.	
	Log N_0	Log $\mathrm{N}_0^{(s)}$	Log N_0	Log $\mathrm{N}_0^{(s)}$	Log N_0	Log $\mathrm{N}_0^{(s)}$	Log N_0	Log $\mathrm{N}_0^{(s)}$
4,500	21·30	10·51	20·80	10·01	20·30	9·51	19·80	9·01
5,000	21·40	11·76	20·90	11·26	20·40	10·76	19·90	10·26
5,500	21·49	12·77	20·99	12·27	20·49	11·78	19·99	11·27
6,000	21·56	13·63	21·07	13·13	20·58	12·64	20·08	12·14
6,500	21·59	14·31	21·13	13·85	20·64	13·37	20·15	12·87
7,000	21·50	14·78	21·13	14·41	20·68	13·97	20·21	13·49
8,000	20·84	15·04	20·74	14·94	20·53	14·74	20·21	14·41
9,000	20·05	14·96	20·03	14·95	19·99	14·90	19·89	14·80
10,000	19·39	14·87	19·38	14·87	19·37	14·85	19·36	14·84
11,000	18·85	14·79	18·85	14·79	18·85	14·79	18·84	14·78
12,000	18·41	14·74	18·41	14·74	18·41	14·74	18·41	14·74
13,000	18·04	14·70	18·04	14·70	18·04	14·70	18·04	14·70
14,000	17·73	14·67	17·73	14·67	17·73	14·67	17·73	14·67

TABLE VIII.—Hydrogen. Calculated numbers of atoms per cm.² down to $\tau = \frac{1}{3}$ capable of absorbing BALMER lines ($\mathrm{N}_0^{(s)}$) for various g and T, assuming 5 per cent. by mass of hydrogen for κ = constant = 100. (N_0 = number of neutral atoms.)

g	10		10^2		10^3		10^4	
T	Log N_0	Log $\mathrm{N}_0^{(s)}$	Log N_0	Log $\mathrm{N}_0^{(s)}$	Log N_0	Log $\mathrm{N}_0^{(s)}$	Log N_0	Log $\mathrm{N}_0^{(s)}$
4,500	20·00	9·21	20·00	9·21	20·00	9·21	20·00	9·21
5,000	20·00	10·36	20·00	10·36	20·00	10·36	20·00	10·36
5,500	19·97	11·25	20·00	11·28	20·00	11·28	20·00	11·28
6,000	19·79	11·86	19·96	12·02	20·00	12·06	20·00	12·06
6,500	19·27	11·99	19·81	12·53	19·97	12·69	20·00	12·72
7,000	18·53	11·82	19·39	12·67	19·85	13·13	19·98	13·26
8,000	17·19	11·39	18·12	12·32	19·12	13·32	19·74	13·94
9,000	16·11	11·02	17·11	12·02	18·10	13·02	19·04	13·96
10,000	15·24	10·72	16·24	11·72	17·24	12·72	18·22	13·71
11,000	14·51	10·45	15·51	11·45	16·51	12·45	17·51	13·45
12,000	13·90	10·23	14·90	11·23	15·90	12·23	16·90	13·23
13,000	13·37	10·04	14·37	11·04	15·37	12·04	16·37	13·04

These curves demonstrate in the clearest possible way the difference between $\kappa \propto$ P and $\kappa =$ constant. Fig. 3 shows that if $\kappa \propto$ P, "absolute magnitude effects" for the BALMER lines are only to be expected at temperatures below A_0. At temperatures above A_0 the curves merge together and $N_0^{(s)}$ is independent of g; at lower temperatures they separate. The horizontal interval between any two curves, say between $g = 10$ and $g = 10^4$, gives the calculated difference of temperature for giants and dwarfs (of these g's) of the same spectral type as judged by the intensity of their hydrogen absorption lines. But if κ is taken as constant, there should be no "absolute magnitude" effect at low temperatures; instead a strong one should set in at high temperatures; the curves of fig. 4 coincide at low temperatures but separate at high temperatures.*

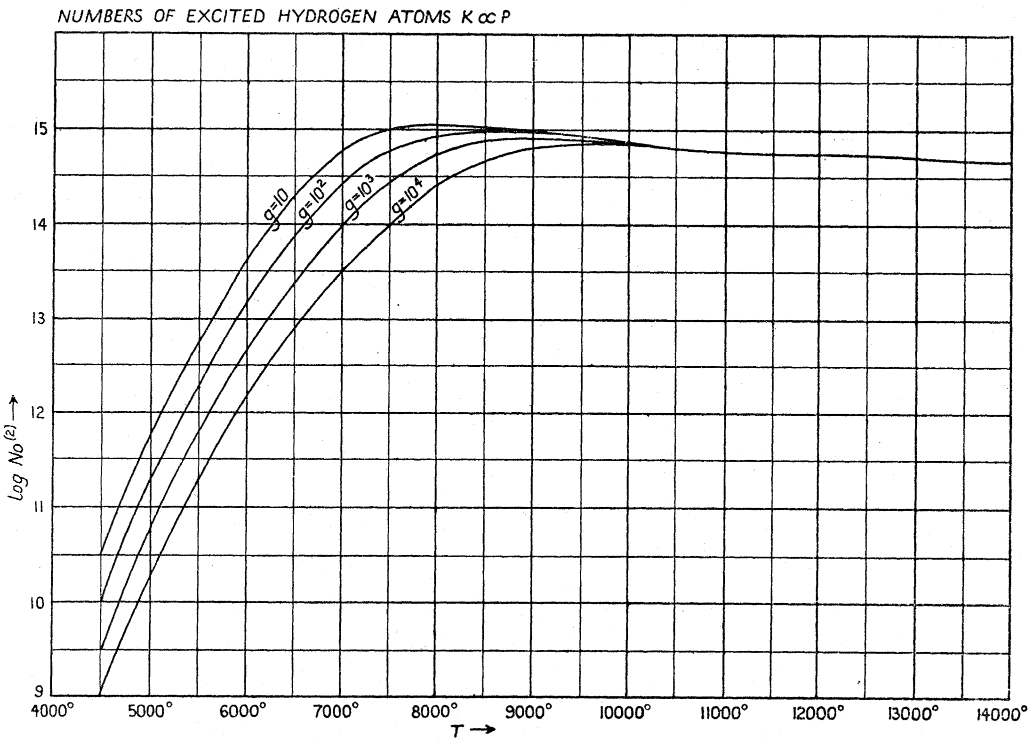


FIG. 3.—Numbers of excited hydrogen atoms (capable of absorbing BALMER lines) above $\tau = \frac{1}{3}$, for various values of temperature T and surface gravity g, calculated for $\varepsilon = 0 \cdot 05$ (proportion by mass) and photospheric opacity $\kappa = 0 \cdot 85\ P/(T/10^4)^{9/2}$, where P = electron pressure. For this law of photospheric opacity the curve show no "absolute magnitude effect" above T = 10,000°. The horizontal distance between any two curves at lower temperatures gives the computed difference of temperature between giant (low g) and dwarf (high g) showing BALMER lines of same strength.

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* The law $\kappa \propto$ P predicts a null absolute magnitude effect at high temperatures *as arising from changes of atmospheric thickness and of ionization;* the two effects neutralise one another. But this does not preclude other effects of g-changes. For example reduced g, which causes reduced P, will cause a decreased Stark effect, which might be used as an absolute magnitude criterion. I owe this remark to discussions with Miss WILLIAMS, of Harvard College Observatory. See "Harvard College Observatory Bulletin," No. 870 (*in press.*)

The observed facts are that hydrogen shows a strong absolute magnitude effect at temperatures below A_0* (increasing intensity accompanying increasing luminosity) and the *c*-effect in classes A and B (increasing sharpness accompanying increasing luminosity). The second effect is well shown in PAYNE and WILLIAM'S line-contours (*loc. cit.*) for stars of type B_5. If following UNSÖLD we take increasing sharpness to indicate a decreasing number of fully-viewed atoms, then the observed facts correspond to a kind of mixture of the effects shown by our two diagrams. At low temperatures, the hypothesis $\kappa \propto P$ is confirmed; the curves for the lower *g*-values lie above the curves for the higher *g*-values, as observed. At high temperatures the reverse effect is shown; observation seems to favour $\kappa =$ constant. This is a point for future investigation. In the past hydrogen lines have not been used as a quantitative criterion for absolute magnitude at low temperatures; and at high temperatures, they have been only partially so used, as there appears to be a correlation of type with absolute magnitude which is hard to disentangle.† But it is quite clear that ultimately the behaviour of the hydrogen lines with *g* and T will afford delicate tests of any theory of the photospheric absorption coefficient.

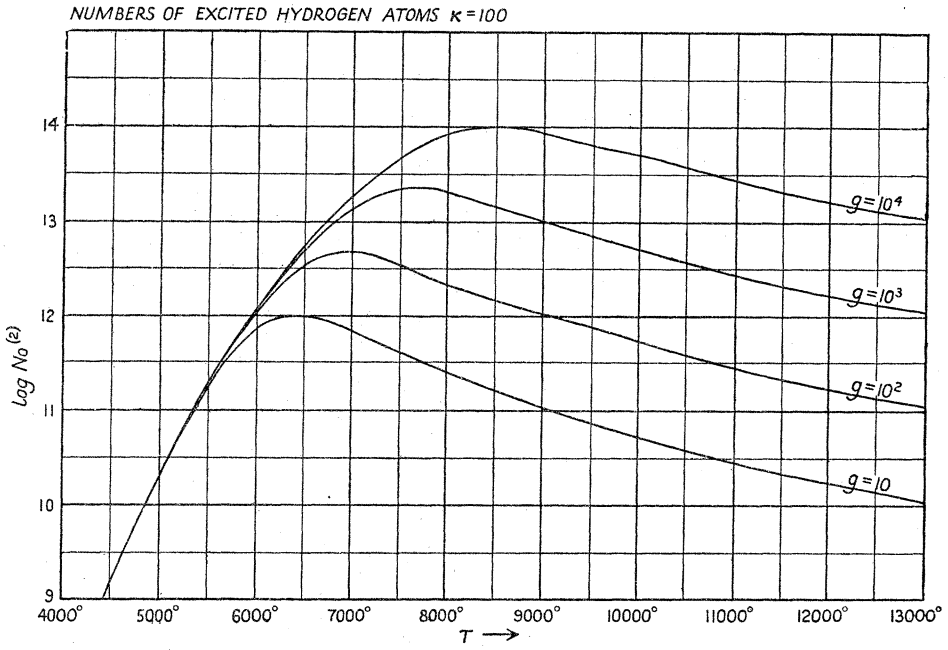

FIG. 4.—Numbers of excited hydrogen atoms (capable of absorbing BALMER lines) above $\tau = \frac{1}{3}$, for various values of temperature T and surface gravity *g*, calculated for $\varepsilon = 0 \cdot 05$ (proportion by mass) and photospheric opacity $\kappa =$ constant $= 100$. These curves are inserted simply for contrast with those of Fig. 3. It will be seen that for $\kappa =$ constant there should be a strong "absolute magnitude" effect at high temperature but none at low temperatures.

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* 'Trans. Internat. Astron. Union,' vol. 1, p. 98 (1922).

† ADAMS and JOY, 'Astrophys. J.,' vol. 56, p. 242 (1922), and vol. 57, p. 294 (1923).

The calculated values of the numbers of fully-viewed excited hydrogen atoms at low temperatures scarcely seem adequate to account for the great strength of the BALMER lines in late-type giants ; ADAMS and RUSSELL* have found that the number of excited hydrogen atoms is much larger at low temperatures than can be accounted for by thermodynamic theory. Perhaps the excited fraction is much larger than is given by the BOLTZMANN factor.

Once again it may be mentioned that the calculated numbers of hydrogen atoms given in our tables and curves do not involve any datum of observation referring to hydrogen in stellar atmospheres ; they are derived from the observed Zn maximum in the case $\kappa \propto \mathrm{P}$, and from an assumed order of magnitude of κ in the case $\kappa =$ constant. Our proportion by mass $\varepsilon = 0\cdot05$ may be of course too small, in view of ROSSELAND'S suggestion of an excess of hydrogen in stellar atmospheres.†

These two examples, Ca and H, will serve to show the fundamental importance of an accurate knowledge of the behaviour and order of magnitude of the photospheric absorption coefficient. A few further examples are given, relating to Mg, Mg^+, He, He^+, Si^+, Si^{++}, Si^{+++}. The numbers of silicon atoms responsible for various lines are shown, calculated for two values of g ($g = 10^3$ and $g = 10^4$), $\varepsilon = 0\cdot05$, and $\alpha = 0\cdot85 \times 10^{18}$. The calculated electron pressures are also shown at certain places on each curve, at the depth $\tau = \frac{1}{3}$. For Mg, He and He^+ ε is taken as $0\cdot05$ for definiteness, but of course the curves require simple vertical displacement to correspond to other values of ε ; for these latter curves g has been taken to be $6\cdot08 \times 10^2$, the value for Capella. See figs. 5 and 6.

16. *Application to Determine the Physical State of the Photospheric Layers.*—Our last application of the determination of photospheric opacity is to calculate the distribution of pressure, temperature and opacity through the photospheric layers of two typical stars. The theory is a slight modification of one previously given,‡ but the approximations now made are of a much more accurate nature.

We have previously calculated the mechanical equilibrium of the layer in which general opacity sets in, taking account of the variation of ionization with depth but ignoring radiation pressure. We now propose to allow for radiation pressure, but we have to pay the penalty in that we are compelled to take a mean value for the ionization. Selective radiation pressure is not allowed for, so that the investigation which follows may be taken to apply to the lower reversing layer and the photospheric layers.

Let p' denote the pressure of radiation at depth h in the photospheric layers, measured from some convenient reference level. Then since a net flux $\pi\mathrm{F}$ traversing a layer $\rho\, dh$ of absorption coefficient κ communicates momentum $\pi\mathrm{F}\, \kappa\rho\, dh/c$ we have

$$\frac{dp'}{dh} = \frac{\kappa\rho}{c}\frac{\pi\mathrm{F}}{}. \qquad (66)$$

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[*] 'Astrophys. J.,' vol. 68, p. 9 (1928).

[†] 'M.N.R.A.S.,' vol. 85, p. 541 (1925).

[‡] 'M.N.R.A.S.,' vol. 85, p. 768 (1925).

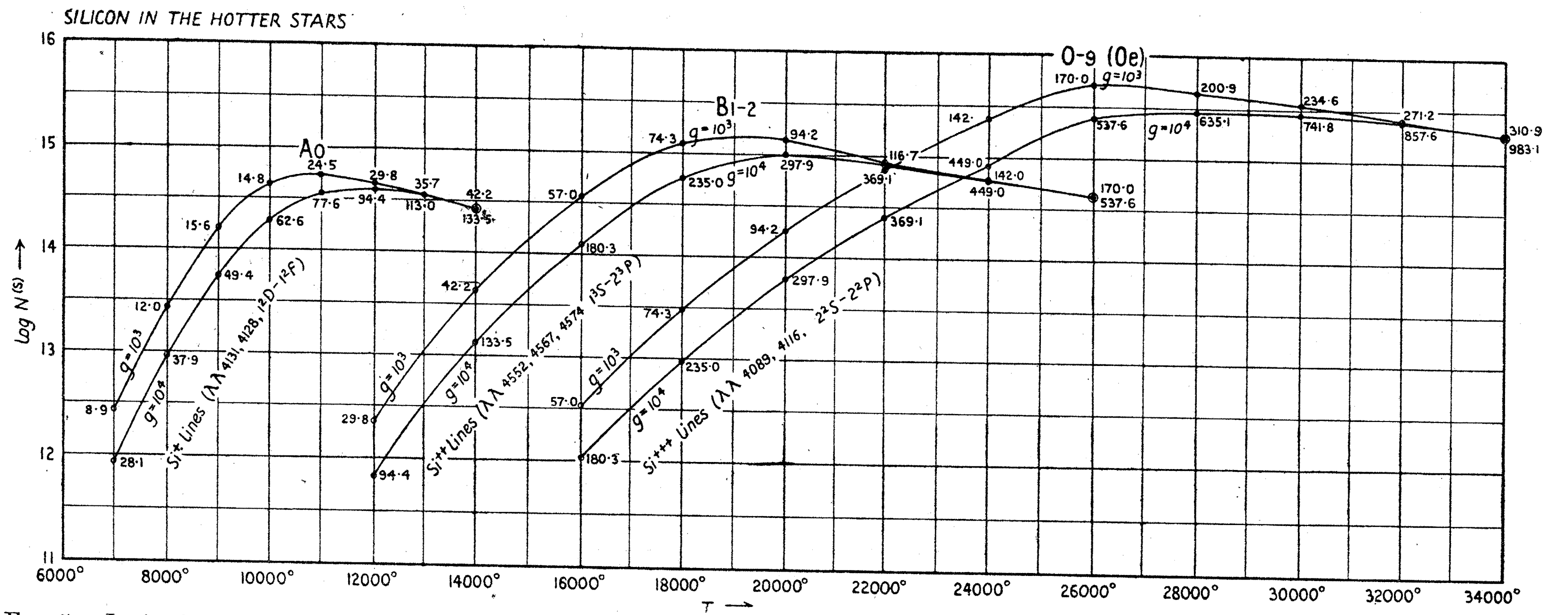

FIG. 5.—*Ionization of silicon in the hotter stars.*—The curves show the calculated numbers $N_1^{(s)}$ of excited Si^+ atoms, $N_2^{(s)}$ of excited Si^{++} atoms, and $N_3^{(s)}$ of excited Si^{+++} atoms per cm.² above $\tau = \frac{1}{3}$, at various temperatures and for two different values of surface gravity of g, $g = 10^3$ and $g = 10^4$ cms. sec.$^{-2}$. Assumed proportion of silicon, 5 per cent. by mass ($\varepsilon = 0 \cdot 05$). The small figures near the curves are the electron pressures at $\tau = \frac{1}{3}$ at the corresponding temperatures, in dynes cm.$^{-2}$. The calculated curves may be compared with Miss PAYNE's estimates of line-intensity plotted in Harvard Circular No. 252 (1924), p. 8. The spectral types marked near the peaks of the curves are the types of observed maxima.

Note.—The descending branches of the curves for Si^+ and Si^{++} are not accurate, as no account has been taken of the onset of the succeeding ionization. The descending branches should really be steeper (see R. H. Fowler, 'Statistical Mechanics,' p. 385).

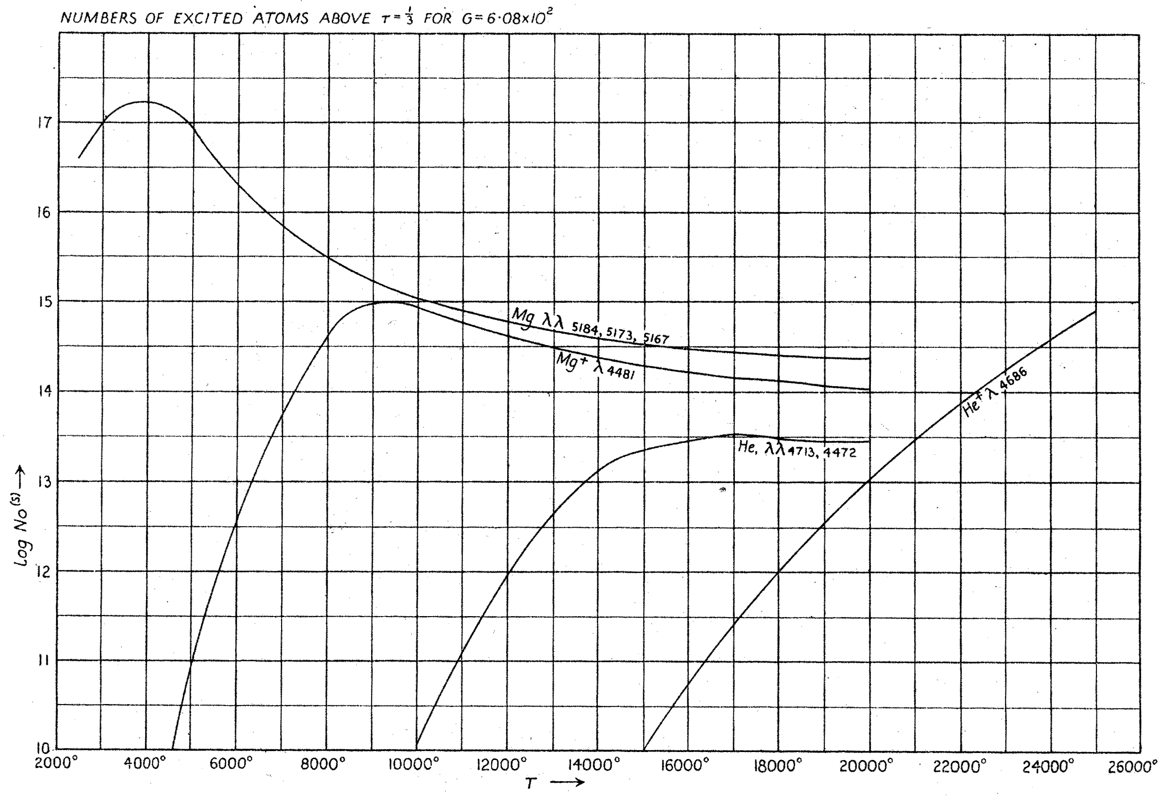

Fig. 6.—Numbers of excited atoms (capable of absorbing certain lines) above $\tau = \frac{1}{3}$, per cm.^{2}, calculated for $g = 6 \cdot 08 \times 10^{-2}$ cm. sec.$^{-2}$. In each case the assumed proportion is 5 per cent. by mass ($\varepsilon = 0 \cdot 05$). Different proportions are obtained by displacing the curves rigidly up or down. (N.B.—Owing to an error, inaccurate statistical weights were employed in the calculation of the curves of this diagram, but this simply corresponds to small vertical displacements, not necessarily the same for each curve). (Second-stage ionization has not been allowed for in the descending branch of the Mg curve).

This is really the definition of p'. To connect it with the temperature we notice that it may be written

$$\frac{dp'}{d\tau} = \frac{\pi \mathrm{F}}{c} = \frac{\sigma \mathrm{T}_1{}^4}{c}.$$

By the solution to Schwarzschild's problem, generalised as in §5, we have by differentiation

$$\frac{d\mathrm{T}^4}{d\tau} = \tfrac{3}{4}\mathrm{T}_1{}^4.$$

Hence

$$\frac{dp'}{d\tau} = \frac{1}{3}\frac{4\sigma}{c}\frac{d\mathrm{T}^4}{d\tau},$$

or integrating

$$p' = \tfrac{1}{3}a\mathrm{T}^4, \quad \ldots \ldots \ldots \ldots \quad (67)$$

provided

$$a = 4\sigma/c.$$

TABLE VIIIA.—Behaviour of Ionized Silicon in Stellar Atmospheres. Calculated for $\alpha = 0\cdot85\ P/(T/10^4)^{9/2}$ and $\varepsilon = 0\cdot05$. (For spectroscopic data here used, see R. H. FOWLER, 'Statistical Mechanics,' pp. 377, 378). P_0 in dynes cm.$^{-2}$.

			Si^+ (λλ 4131, 4128)				Si^{++} (λλ 4552, 4567, 4574)				Si^{+++} (λλ 4089, 4116)			
T	P_0 $g = 10^3$	P_0 $g = 10^4$	$g = 10^3$		$g = 10^4$		$g = 10^3$		$g = 10^4$		$g = 10^3$		$g = 10^4$	
			Log N_1	Log $N_1^{(s)}$	Log N_1	Log $N_1^{(s)}$	Log N_2	Log $N_2^{(s)}$	Log N_2	Log $N_2^{(s)}$	Log N_3	Log $N_3^{(s)}$	Log N_3	Log $N_3^{(s)}$
7,000	8·9	28·1	18·28	12·44	18·78	11·94	—	—	—	—	—	—	—	—
8,000	12·0	37·9	18·40	13·44	18·90	12·95	—	—	—	—	—	—	—	—
9,000	15·6	49·4	18·46	14·20	19·00	13·73	—	—	—	—	—	—	—	—
10,000	19·8	62·6	18·35	14·63	19·00	14·28	—	—	—	—	—	—	—	—
11,000	24·5	77·6	18·00	14·72	18·83	14·55	—	—	—	—	—	—	—	—
12,000	29·8	94·4	17·54	14·65	18·49	14·59	19·80	12·33	19·30	11·83	—	—	—	—
13,000	35·7	113·0	17·11	14·54	18·10	14·52	—	—	—	—	—	—	—	—
14,000	42·2	133·5	—	—	17·73	14·41	19·95	13·61	19·45	13·11	—	—	—	—
16,000	57·0	180·3	—	—	—	—	20·05	14·55	19·57	14·08	20·08	12·53	19·58	12·03
18,000	74·3	235·0	—	—	—	—	19·93	15·10	19·57	14·75	20·20	13·49	19·70	12·99
20,000	94·2	297·9	—	—	—	—	19·44	15·14	19·30	15·00	20·30	14·25	19·80	13·76
22,000	116·7	369·1	—	—	—	—	18·82	14·95	18·79	14·92	20·37	14·87	19·89	14·39
24,000	142·0	449·0	—	—	—	—	18·27	14·77	18·27	14·76	20·39	15·36	19·94	14·91
26,000	170·0	537·6	—	—	—	—	17·80	14·61	17·80	14·60	20·36	15·71	20·05	15·40
28,000	200·9	635·1	—	—	—	—	—	—	—	—	19·96	15·64	19·77	15·45
30,000	234·6	741·8	—	—	—	—	—	—	—	—	19·58	15·55	19·49	15·46
32,000	271·2	857·6	—	—	—	—	—	—	—	—	19·18	15·40	19·16	15·38
34,000	310·9	983·1	—	—	—	—	—	—	—	—	18·83	15·28	18·82	15·27

TABLE VIIIB.—Behaviour of Normal Excited C^+ Atoms in Stellar Atmospheres calculated for $\kappa = 0{\cdot}85\ P/(T/10^4)^{9/2}$. The values of $N_1^{(s)}$ should give the strength of the line λ 4267, $1\,{}^2D—1\,{}^2F$. Observed maximum is at type B_3. $g = 10^4$ cms. sec.$^{-2}$.

T	P_0 dynes cm.$^{-2}$	Log N_1	Log $N_1^{(s)}$
12,000	94·38	19·67	12·34
14,000	133·51	19·77	13·53
16,000	180·30	19·64	14·21
18,000	235·01	19·19	14·38
20,000	297·89	18·66	14·36
22,000	369·13	18·20	14·31
24,000	448·97	17·82	14·27

TABLE VIIIC.—Behaviour of Normal and Excited N^+ Atoms in Stellar Atmospheres calculated for $L = 0{\cdot}85\,P/(T/10^4)^{9/2}$. The values of $N_1^{(s)}$ should give the strength of the line λ 3995, $1\,{}^1P'—1\,{}^1P$. The observed maximum is at type B_{3-5}. $g = 10^4$ cms. sec.$^{-2}$.

T	P_0 dynes cm.$^{-2}$	Log N_1	Log $N_1^{(s)}$
14,000	133·51	19·76	12·65
16,000	180·30	19·85	13·57
18,000	235·01	19·80	14·16
20,000	297·89	19·47	14·35
22,000	369·13	18·99	14·29
24,000	448·97	18·53	14·18
26,000	537·56	18·13	14·08

This is the usual definition of the constant a, and (67) agrees with the formula for p' for thermo-dynamic equilibrium. Thus (67) holds to the extent to which $T^4 = \frac{1}{2}T_1{}^4(1 + \frac{3}{2}\tau)$ is an accurate solution of SCHWARZSCHILD'S problem.

The equation of mechanical equilibrium is

$$\frac{dp}{dh} + \frac{dp'}{dh} = g\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Dividing (68) by (66),

$$1 + \frac{dp}{dp'} = \frac{cg}{\pi\kappa F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Now if M is the mass of the star, r_1 its radius, L its bolometric luminosity, $g = GM/r_1{}^2$ and $\pi F = L/4\pi r_1{}^2$. Hence $g/\pi F = 4\pi GM/L$. Further

$$\kappa = \frac{\alpha P}{T^{9/2}} = \alpha P\left(\frac{\frac{1}{3}a}{p'}\right)^{9/8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

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We shall now suppose there are on the average $\bar{x}$ free electrons per atom in the photospheric layers.* Then

$$\frac{\mathrm{P}}{p} = \frac{\bar{x}}{1+\bar{x}}.$$

Then (69) becomes

$$1 + \frac{dp}{dp'} = \frac{4\pi c \mathrm{GM}}{\mathrm{L}} \frac{(1+\bar{x})/\bar{x}}{\alpha\,(\frac{1}{3}a)^{9/8}} \frac{p'^{\,9/8}}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Make the substitution†

$$p = u\,\frac{p'^{\,17/16}}{p_1'^{\,1/16}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

where

$$p_1' = \tfrac{1}{3}a\mathrm{T}_1{}^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Then (71) becomes

$$\left(\frac{p_1'}{p'}\right)^{1/16} + p'\frac{du}{dp'} + \frac{17}{16}u = \frac{4\pi c\mathrm{GM}}{\mathrm{L}} \frac{(1+\bar{x})/\bar{x}}{\alpha\,(\frac{1}{3}a)^{9/8}} \frac{p_1'^{\,1/8}}{u}$$

$$= \left[\frac{4\pi c\mathrm{GM}}{\mathrm{L}} \frac{(1+\bar{x})/\bar{x}}{\alpha\,(\frac{1}{3}a)} \mathrm{T}_1{}^{\frac{1}{2}}\right]\frac{1}{u}. \quad . \quad . \quad . \quad . \quad (74)$$

Denote the coefficient of u^{-1} on the right-hand side by A.

The term $(p_1'/p')^{1/16}$ varies as $\mathrm{T}^{-\frac{1}{4}}$ and is equal to unity for $\mathrm{T} = \mathrm{T}_1$. This variation is so slight for the range of values of T we shall have to consider that we shall ignore it, replacing the term by unity. Equation (74) is then soluble. It may be written

$$\frac{dp'}{p'} = \frac{u\,du}{\mathrm{A} - u - \frac{17}{16}u^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Writing

$$\mathrm{A} - u - \tfrac{17}{16}u^2 \equiv -\tfrac{17}{16}(u - u_1)(u + u_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

we have on integration‡

$$\log\frac{p'}{p'_0} = -\frac{16/17}{u_1 + u_2}\left[u_1\log\left(1 - \frac{u}{u_1}\right) + u_2\log\left(1 + \frac{u}{u_2}\right)\right], \quad . \quad . \quad . \quad (77)$$

since $u = 0$ when $p' = p_0' = \frac{1}{3}a\mathrm{T}_0{}^4$.

We have also by the solution to SCHWARZSCHILD's problem

$$p'/p_0' = 1 + \tfrac{3}{2}\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

We see that as τ increases, and accordingly p'/p_0' increases, u rapidly increases§ and approaches the limit u_1.‖ Hence by (72) the ratio of gas-pressure to radiation-pressure, p/p', slowly increases in the deeper layers, being equal to $u_1(p'/p_1')^{1/16}$. Thus p/p' ultimately increases as $\mathrm{T}^{\frac{1}{4}}$.

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* The equation is rather intractable if we try to calculate the variation of x through the layer.

† The physical meaning of u is most readily understood by noting that at the level $\mathrm{T} = \mathrm{T}_1$, where $p' = p'_1$, u is equal to the ratio of gas-pressure to radiation pressure.

‡ WOLTJER, B.A.N., vol. 2, p. 171 (No. 66) (1924).

§ JEANS, 'M.N.R.A.S.,' vol. 85, p. 199 (1925).

‖ This argument is due in principle to JEANS, whose analysis we have slightly modified.

Equations (77), (78) and (72) determine p', τ and p for given u. p' determines the temperature T. It remains to determine the heights to which these values of u refer. From (66) we have, on putting $\rho = p/(\mathrm{R}/\mu)\mathrm{T}$ and inserting the formula for κ,

$$\frac{dp'}{dh} = \frac{\alpha\bar{x}}{1+\bar{x}}\left(\frac{\frac{1}{3}a}{p'}\right)^{9/8}\frac{p^2}{(\mathrm{R}/\mu)\,\mathrm{T}}\,\frac{g\mathrm{L}}{4\pi c\mathrm{GM}}.$$

Hence using (72)

$$\frac{1}{\mathrm{T}}\frac{d\mathrm{T}}{dh} = \frac{1}{4}\frac{1}{p'}\frac{dp'}{dh} = \frac{\frac{1}{4}\alpha\bar{x}}{1+\bar{x}}\,\frac{(\frac{1}{3}a)^{9/8}}{p'^{\,17/8}}\,\frac{p^2}{(\mathrm{R}/\mu)\,\mathrm{T}}\,\frac{g\mathrm{L}}{4\pi c\mathrm{GM}}$$

$$= \frac{\frac{1}{4}u^2 g}{\mathrm{A}\,(\mathrm{R}/\mu)\,\mathrm{T}}.$$

The value of this formula is that T cancels from each side and we obtain

$$\frac{d\mathrm{T}}{dh} = \frac{\frac{1}{4}g}{\mathrm{A}\,(\mathrm{R}/\mu)}\,u^2,$$

or by (76)

$$\frac{d\mathrm{T}}{dh} = \frac{\frac{1}{4}g}{\mathrm{R}/\mu}\cdot\frac{u^2}{(17/16)\,u_1u_2}. \qquad (79)$$

This is an exact formula for the temperature gradient involving no approximations except the use of $\bar{x}$ instead of variable x. We observe that as $u \rightarrow u_1$,

$$\frac{d\mathrm{T}}{dh} \rightarrow \frac{\frac{1}{4}g}{\mathrm{R}/\mu}\cdot\frac{u_1}{(17/16)\,u_2}.$$

The temperature gradient approaches a definite limit. Since $u_2 - u_1 = 16/17$, the value of this limit is

$$\left(\frac{d\mathrm{T}}{dh}\right)_\infty = \frac{\frac{1}{4}g}{\mathrm{R}/\mu}\cdot\frac{1}{(17/16) + u_1^{-1}}.$$

Thus provided u_1 is not too small, the temperature gradient at great depths is roughly independent of A, and so of α and L/M. It can never exceed $(4/17)\,g/(\mathrm{R}/\mu)$.

To obtain the height for given u, we write (79) in the form

$$\frac{4}{\mathrm{T}}\frac{d\mathrm{T}}{dh} = \frac{1}{p'}\frac{dp'}{dh} = \frac{g}{(\mathrm{R}/\mu)\,\mathrm{T}}\,\frac{u^2}{(17/16)\,u_1u_2},$$

and substitute for dp'/p' from (75). The result is

$$\frac{g\,dh}{(\mathrm{R}/\mu)\,\mathrm{T}} = -\frac{u_1u_2\,du}{u\,(u-u_1)\,(u+u_2)}, \qquad (80)$$

To avoid complicated integrals it is convenient to take a mean value for T on the left-hand side here. Denoting it by $\overline{\mathrm{T}}$ we have

$$h - h_0 = \frac{(\mathrm{R}/\mu)\,\overline{\mathrm{T}}}{g}\left[\log u - \frac{u_2}{u_1+u_2}\log\left(1-\frac{u}{u_1}\right) - \frac{u_1}{u_1+u_2}\log\left(1+\frac{u}{u_2}\right)\right] \quad (81)$$

where h_0 is arbitrary.

Tables IX and X show the physical conditions of the photospheric layers of the Sun and Capella (bright component) calculated from the above formulæ. The value T required in (81) has been taken to be $\frac{1}{2}(T_0 + T)$. Actually a sort of harmonic mean is

TABLE IX.—The Sun ($g = 2 \cdot 74 \times 10^4$ cm. sec.$^{-2}$, $T_1 = 5740°$). Physical state of photospheric layers. $u_1 = 29 \cdot 0$, $u_2 = 29 \cdot 9$, $1/(1 + u_1) = 0 \cdot 0333$.

u	Optical thickness τ	Radiation pressure p' dynes cm.$^{-2}$	Gas-pressure p dynes cm.$^{-2}$	Temperature T degrees	Temperature gradient dT/dh degrees km^{-1}	Depth h kms.	Density ρ gram cm.$^{-3}$	Absorption coefficient κ
0	0	1·386	0	4,830	0	$-\infty$	0	0
5	0·009	1·405	6·72	4,840	4·5	11·9	$3 \cdot 36 \times 10^{-10}$	74·6
10	0·040	1·468	14·11	4,900	18·0	17·5	$6 \cdot 95 \times 10^{-10}$	149
15	0·103	1·600	23·2	5,000	40·5	21·4	$11 \cdot 2 \times 10^{-10}$	222
20	0·231	1·862	36·3	5,200	72·0	25·3	$16 \cdot 85 \times 10^{-10}$	293
25	0·582	2·595	64·5	5,650	112·7	31·1	$27 \cdot 7 \times 10^{-10}$	359
27	1·027	3·515	96·3	6,090	131·3	35·8	$38 \cdot 3 \times 10^{-10}$	381
28	1·650	4·81	139·6	6,590	141·1	40·7	$51 \cdot 0 \times 10^{-10}$	388
28·3	2·054	5·65	168·0	6,860	144·2	43·3	$59 \cdot 1 \times 10^{-10}$	390
28·6	2·854	7·30	222	7,310	147·3	47·7	$73 \cdot 3 \times 10^{-10}$	385
28·8	4·185	10·09	315	7,920	149·2	53·6	$96 \cdot 4 \times 10^{-10}$	380
28·9	6·03	13·91	444	8,540	150·3	59·8	126×10^{-10}	375
28·95	8·57	19·17	625	9,300	151·0	67·0	613×10^{-10}	367

TABLE X.—Capella (bright component) ($g = 6 \cdot 08 \times 10^2$ cm. sec.$^{-2}$, $T_1 = 5200°$). Physical state of photospheric layers. $u_1 = 4 \cdot 80$, $u_2 = 5 \cdot 74$, $1/(1 + u_1) = 0 \cdot 172$.

u	Optical thickness τ	Radiation pressure p' dynes cm.$^{-2}$	Gas-pressure p dynes cm.$^{-2}$	Temperature T degrees	Temperature gradient dT/dh degrees km.$^{-1}$	Depth h kms.	Density ρ gram cm.$^{-3}$	Absorption coefficient κ
0	0	0·932	0	4,370	0	$-\infty$	0	0
1	0·013	0·950	0·910	4,390	0·13	16	$0 \cdot 501 \times 10^{-10}$	15·7
2	0·055	1·010	1·94	4,460	0·50	255	$1 \cdot 05 \times 10^{-10}$	31·2
3	0·153	1·146	3·33	4,600	1·13	440	$1 \cdot 75 \times 10^{-10}$	46·5
3·5	0·249	1·281	4·37	4,730	1·54	540	$2 \cdot 24 \times 10^{-10}$	53·8
4·0	0·443	1·553	6·14	4,960	2·01	672	$3 \cdot 00 \times 10^{-10}$	60·9
4·2	0·559	1·715	7·16	5,090	2·22	745	$3 \cdot 40 \times 10^{-10}$	63·7
4·4	0·780	2·025	8·95	5,300	2·43	844	$4 \cdot 08 \times 10^{-10}$	65·8
4·6	1·261	2·695	12·70	5,700	2·66	1,023	$5 \cdot 38 \times 10^{-10}$	67·7
4·7	1·909	3·61	17·70	6,120	2·78	1,205	$6 \cdot 99 \times 10^{-10}$	67·9
4·75	2·799	4·85	24·4	6,590	2·83	1,403	$8 \cdot 95 \times 10^{-10}$	67·1
4·78	4·462	7·17	37·2	7,270	2·87	1,690	$12 \cdot 40 \times 10^{-10}$	66·0
4·79	6·247	9·68	51·5	7,830	2·88	1,928	$15 \cdot 9 \times 10^{-10}$	65·3

required. In fact once the temperatures have been obtained from (77) and the gradients from (79), the heights could be found accurately by quadrature. This did not seem worth while, so that the heights given in the tables, though approximately correct, do not exactly tally with the tabulated temperature-gradients, which are accurate. The tables are calculated for an assumed molecular weight $\mu = 20$, corresponding to once-ionized calcium, which would seem to be typical for the Sun and Capella ; $\bar{x}$ is taken to be unity. The values of the absorption coefficient κ at each level are also calculated. The whole calculation is based on the value $\alpha = 0 \cdot 85 \times 10^{18}$.

It will be seen that the tables reproduce, as is only natural, values for κ of the same order as found by the method of maxima. The temperature gradient in the photosphere of Capella never exceeds 3° per kilometre ; that in the Sun rises to 150° per kilometre. The heights tabulated contain an arbitrary additive constant. We see that for the Sun, the layer $\tau = 0 \cdot 01$ to $\tau = 1 \cdot 03$ is contained in 24 km. ; to $\tau = 4$, which is about the furthest we can see at the centre of the disc ($e^{-4} = 0 \cdot 018$), in 42 km. ; for Capella, from $\tau = 0 \cdot 01$ to $\tau = 0 \cdot 78$ is contained in 830 km. ; to $\tau = 4$, in about 1600 km. The distances are roughly in the ratios of the g-values.

These tables do not take into account the selective radiation pressure which rises into importance in the upper reversing layer. Consequently the actual thickness of the reversing layer will be greater than the values found.

The ratio of radiation-pressure to total pressure at the depth where $T = T_1$ is equal to $1/(1 + u_1)$. This is tabulated at the head of each table.

17. *Imperfections of the Method.*—Both the theory of the present method and its numerical applications must for various reasons be regarded as provisional. The theory itself is open to criticism on a number of grounds, which may be divided into (*a*) mathematical, (*b*) physical.

(*a*) The formula for the depth of the equivalent photosphere, $\tau = \frac{4}{3} r/(r + 1)$ is derived only for a special model, and then applied to cases in which the assumption of the special model no longer hold. It assumes that lines are formed by strict monochromatic scattering by atoms embedded in a gaseous medium contributing a continuous background. In deriving the formula, κ is assumed to bear a constant ratio to s_ν. We then apply it to the case in which $\kappa \propto P$ whilst s_ν varies with P according to a different law determined by the variation of ionization with P. We assume that in the actual atmosphere study may be confined to the atoms contained in the uppermost τ of optical thickness. However, all these assumptions are equivalent to the assumption that failing mathematically rigorous solutions of the general equations we are entitled to introduce averaging processes. The averaging processes here introduced are provisional, to be replaced in due course by progress in the mathematical analysis.

(*b*) The method of deriving the absorption coefficient of the photospheric layers seems at first sight to be independent of any assumption about their chemical composition ; factors we have called ε disappear. We have, however, implicitly assumed that something is known about the mean ionization potential of the material. For example in analysing the Zn maximum we have assumed that in solar type stars the material may be assumed to be on the whole just once ionized; alternatively, we have assumed that the mean ionization potential of the material is of the order of the comparatively low first-stage ionization potential of the metals. The results would require re-calculation should it be found that there is, in solar stars, a considerable proportion of neutral atoms, *e.g.*, atoms of hydrogen.* The average ionization-potential would then exceed that of the metals.

Neither of these classes of imperfections appear, however, to destroy the general principles of the method. Accumulating measures of line-contours will ultimately permit our provisional assumptions to be replaced by more accurate ones. The details of the application of the method are flexible; the method itself appears to the author to remain valid. The advantage of the method of maxima is that it avoids any appeal to a knowledge of the exact formula for the atomic scattering coefficient. In fact if the chemical composition factors ε could be assumed known, the atomic scattering coefficients could themselves be determined from observed line-contours. Probably a complete knowledge of all line-contours in a sufficient number of stars of known T and g would provide more equations than unknowns—the unknowns being the ε's and the atomic s_ν's. After that the method could be applied to determine g for stars of known T but unknown g.

An important need at the moment is a mathematical treatment of the effect of selective radiation-pressure in the transition region—the reversing layer. This would in effect enable us to calculate effective gravity in terms of actual gravity and the atomic scattering coefficients. The various unknowns will be entangled with one another in increased complication. But this simply lends an increased attractiveness to the whole subject.

18. *Summary.*—(1) Formulæ are presented for calculating the net outward flux, at any depth, in material of absorption coefficient κ_ν and scattering coefficient s_ν, arbitrary functions of frequency ν in radiative equilibrium.

(2) The temperature distribution is of the type $T^4 = \frac{1}{2}T_1^4(1 + \frac{3}{2}\tau)$, where T_1 is the effective temperature, provided τ the optical thickness is calculated as the "ROSSELAND mean" of the sum of the scattering and absorption coefficients. The boundary temperature may be affected by the behaviour of κ_ν/s_ν as a function of ν.

(3) The results apply generally to the formation of absorption lines by monochromatic scattering as modified by collisions, in the presence of a continuous spectrum formed in any way by true absorption. It is shown that an intensity-ratio r at a point of an absorption line contour corresponds to simple SCHUSTER scattering by an otherwise transparent medium if the "photospheric radiating surface" is placed at an optical depth $\tau = \frac{4}{3}r/(r+1)$. Study of the atoms producing an absorption line of given r can be confined to the uppermost τ of optical thickness.

(4) If a column of equivalent "fully-viewed" N atoms produce an intensity-ratio r, the line is widest for given r when N is a maximum for given τ.

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* I owe this remark in principle to discussions with Mr. W. H. McCrea, of Trinity College, Cambridge.

(5) The number of atoms N may be calculated as a function of g, T and P_0, where g is the surface gravity, T the temperature and P_0 the pressure of free electrons at depth τ. P_0 can only be calculated when the absolute value of the absorption coefficient of photospheric material is known.

(6) An observed temperature $T_{\text{max.}}$ of maximum width of an absorption line for given r determines the electron-pressure P_0 at the base of the column. Consideration of mechanical equilibrium determines the amount of matter in the column of given τ, and hence the absolute value of the absorption coefficient is found.

(7) The absorption coefficient for photospheric material, determined from the observed maximum of Zn lines in G-type stars, is found to be given by the formula

$$\frac{0 \cdot 85\, P}{(T/10^4)^{9/2}}$$

where P is the electron pressure in dynes cm.$^{-2}$. This is a revised value, and the two determinations from the Sun and Capella, corresponding to giant and dwarf conditions, are in good agreement.

(8) Revised calculations from the observed maximum of BALMER lines in A-type stars according to the latest determinations of Miss PAYNE and Miss WILLIAMS are in fair agreement with the above value of the opacity. Observed variations of $T_{\text{max.}}$ with r are accounted for.

(9) At depth $\tau = \frac{1}{3}$, the photospheric absorption coefficients in the Sun and Capella (bright component) are about 300 and 60 respectively.

(10) The photospheric absorption coefficient derived from the Zn maximum is applied to calculate the behaviour of the lines of Ca and Ca^+ in stars of low temperature. Comparison is made with the observations of PAYNE and HOGG reduced by UNSÖLD'S method.

(11) The behaviour of the BALMER lines is calculated. It is shown that with the law $\kappa \propto P$ for photospheric opacity there should be a strong " absolute magnitude effect " at low temperatures, as is observed, but with the law $\kappa =$ constant there should be an absolute magnitude effect at high temperatures but not at low temperatures. It is inferred that photospheric opacity cannot be independent of density, which is in accordance with the attribution of photospheric opacity to the photo-electric effect.

(12) The photospheric opacity determination makes possible a detailed calculation of the physical state of the photospheric layers of a star. Formulæ are derived for the general case, and applied to the Sun and Capella. The maximum temperature gradient in Capella comes out as 3° per kilometre, that in the sun as 150° per kilometre. The thicknesses of the reversing layer are calculated, neglecting selective radiation pressure, and are found to be included within some 20 km. for the Sun, and some 800 km. for Capella. Tables of pressures, temperatures, temperature gradients and values of absorption coefficients are given for various heights.